Black holes with linear dilaton asymptotic and integrability

D.V. Gal'tsov

XXII LOMONOSOV CONFERENCE ON ELEMENTARY PARTICLE PHYSICS
Moscow State University, Moscow, August 21-27, 2025

Based on

[1] D. Gal'tsov and A. Kulitskii, "Petrov types, separability, and generalized photon surfaces of supergravity black holes," Phys. Rev. D **110**, no.12, 124008 (2024) [arXiv:2409.13324].

[2] D. Gal'tsov and R.Karsanov,"Gauged supergravities: Solutions with a Killing tensor," Phys. Rev. D **111**, no.10, 104011 (2025) [arXiv:2503.06589 [gr-qc]].

[3] D.G. and R. Karsanov, work in progress

Prelude: What is black hole?

- Black hole is a globally defined object, interpolating between a regular horizon and infinity
- Standard assumption about infinity is asymptotic flatness (AF) implying spherical topology
- Adding cosmological constant gives rise to ADS black holes with spherical, flat or hyperbolic topology
- Less known are are black holes with asymptotic of *linear dilaton* (ALD), which exist in theories with vector field(s) and dilaton (supergravities)
- Their existence is related to existence of *exact* supersymmetric backgrounds in string theory with linearly growing (in certain coordinates) dilaton field
- They give rise to new holographic models
- Can they have any astrophysical significance?
- What is generic classical ALD black hole? (this talk)

History

- In **S.B. Giddings and A.Strominger, "Dynamics of extremal black holes," Phys. Rev. D 46, 627-637 (1992) [arXiv:hep-th/9202004 [hep-th]]** it was noticed that the near horizon region of an extremal AF dilaton black holes can be considered as a background of some two-dimensional quantum theory
- In 2002 it was realised that this limit gives rise to an exact solution of Einstein-Maxwell-dilaton-axion (EMDA) theory which can be interpreted as non-asymptotically flat non-ADS black hole: **[0] G.Clement, D.Gal'tsov and C.Leygnac, "Linear dilaton black holes," Phys. Rev. D 67, 024012 (2003) [arXiv:hep-th/0208225 [hep-th]]**.
- Rotating version of ALD black hole also was constructed there using Sp(4,R) sigma-model previously developed for EMDA gravity in **D.V. Galtsov and O.V. Kechkin, "Ehlers-Harrison type transformations in dilaton - axion gravity," Phys. Rev. D 50, 7394-7399 (1994) [arXiv:hep-th/9407155 [hep-th]]**
- It was demonstrated that Brown-York mass and angular momentum (Hamiltonian formalism in spacetimes with boundaries) confirm the First law of blak hole thermodynamics

Plan of the talk

- EMDA equations
- Linear dilaton (LD) background
- Asymtoptically LD black hole
- Rotating ALD black holes
- Generalized Lense-Thirring
- Black hole bomb and scalarization
- Constrained Benenti-Francaviglia (BF) ansatz and integrability
- Axidilaton
- Polynomial structure of BF functions
- General ALD solution
- Interpretation of parameters
- Misner string and fluxes
- Conclusions

EMDA equations

- Einstein-Maxwell-dilaton-axion theory is a consistent truncation of $\mathcal{N} = 4$ supergravity with one vector field

$$S = \frac{1}{16\pi} \int \left(-R + 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \sqrt{-g}\, d^4x,$$

where R is a scalar curvature, ϕ and κ are the scalar dilaton and axion fields, and $F_{\mu\nu}$ is the Maxwell tensor.

- Einstein equations, following from this action are given by

$$R_{\mu\nu} = 2\phi_\mu \phi_\nu + \frac{1}{2} e^{4\phi} \kappa_\mu \kappa_\nu + e^{-2\phi} T^{em}{}_\mu{}^\nu,$$

where

$$T^{em}{}_\mu{}^\nu = 2F_{\mu\lambda} F^{\lambda\nu} + \frac{1}{2} \delta_\mu^\nu F_{\alpha\beta} F^{\alpha\beta}$$

is the standard Maxwell energy-momentum tensor.

- Static spherically symmetric spacetime

$$ds^2_{LD} = fdt^2 - f^{-1}dr^2 - r_0 r(d\theta^2 + \sin^2\theta d\varphi^2)$$

supported by the following dilaton and Maxwell fields,

$$f = e^{2\phi} = \frac{r}{r_0}, \quad F = \frac{1}{\sqrt{2}r_0} dr \wedge dt,$$

is an exact solution of the EMDA equations [0].

- Its existence is related to an exact supersymmetric solution of the string theory
- In these coordinates the dilaton exponential is linearly growing as $r \to \infty$ as well radius of two-spheres $r = \mathrm{const}$ at infinity
- Parameter r_0 defines the electric field strength

Asymptotically LD black hole [0]

- Replacing the function

$$f \to \frac{r-2m}{r_0}$$

in the metric, but not in the dilaton exponential, $e^{2\phi} = \frac{r}{r_0}$, we obtain again an exact solution of EMDA theory with a regular horizon at $r = 2m$

- It describes a regular neutral ALD black hole whose physical mass was computed using the Brown-York procedure:

$$\mathcal{M} = m/2$$

- The Hawking temperature obtained from the absence of a conical singularity in the Euclideanized solution is

$$T = 1/(4\pi r_0),$$

and the entropy satisfying the first law

$$d\mathcal{M} = TdS$$

is $S = 2\pi r_0 m$ (r_0 is a background parameter and should not be varied)

Rotating ALD black hole [0]

- The rotating generalization was obtained using sigma-model generating technique

$$ds^2 = \frac{r^2 - 2mr + a^2}{r_0 r} dt^2 - r_0 r \left[\frac{dr^2}{r^2 - 2mr + a^2} + d\theta^2 + \sin^2\theta \left(d\varphi - \frac{a}{r_0 r} dt \right)^2 \right]$$

The vector field and axidilaton are

$$A = \frac{r^2 + a^2\cos^2\theta}{r_0 r} dt + a \sin^2\theta \, d\varphi, \quad z = \kappa + i e^{-2i\phi} = \frac{i r_0}{r - i a \cos\theta}$$

- It has two horizons at $r_\pm = m \pm \sqrt{m^2 - a^2}$. The Brown-York mass and angular momentum $\mathcal{M} = m/2$, $J = a r_0/2$, angular velocity of the event horizon $\Omega_+ = a/r_0 r_+$ and the Hawking temperature

$$T = \frac{r_+ - m}{2\pi r_0 r_+}$$

satisfy the first law $d\mathcal{M} = TdS + \Omega_+ dJ$

Generalized Lense-Thirring

- Lack of spherical symmetry, RLDBH possesses an irreducible Killing tensor $K = \partial_\theta^2 + \partial_\varphi^2 / \sin^2\theta$, so the corresponding Hamilton-Jacoby equations are integrable. This KT is slice-reducible both in foliations of codimension one ($r = const$ and $\theta = const$) (and hence representable in Benenti-Francaviglia form) and with respect to a foliation of a codimension-two surfaces ($t = const, r = const$) which is a property of *generalized Lense-Thirring* metrics (Visser (2020), Kubiznak et al. (2021-2025)).
- Original LT metric can be obtained linearizing Kerr as

$$ds^2 = fdt^2 - 2a\sin^2\theta(f-1)dtd\varphi - dr^2/f - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

with $f = 1 - 2m/r$. This is only an approximate solution of vacuum Einstein equations, which inherits Kerr KT also as approximate quantity. But if one performs squaring in (t, φ) sector, $ds^2 = fdt^2 - dr^2/f - r^2\sin^2\theta\left(d\varphi - adt(f-1)/r^2\right)^2 - r^2d\theta^2$, the above Killing tensor becomes exact. This metric was called *generalized LT* it has a sequence of multidimensional extensions possessing not only rank two KT tensor, but a tower of higher rank KT. Our RLDBH can be put in this form setting $rr_0 \to r^2$

- Radial potential in the Klein-Gordon equation for a massive particle in the RLDBH is

$$V = \frac{\mu^2 r \Delta}{r_0 r_+^2} + \frac{\Delta l(l+1)}{(r_0 r_+)^2} - \frac{r^2}{r_+^2}\left(\omega - \frac{ma}{r_0 r}\right)^2,$$

where μ is particle mass and l, m are usual spherical quantum numbers. On the horizon $\Delta = 0$ it stabilizes at $V = k^2 = (\omega - m\Omega_+)^2$, so modes with $\omega > 0,\ k < 0$ are superradiant.

- At infinity, the potential is growing for massive modes and masseles modes satisfying $\omega r_0 < l + 1/2$. Therefore, all superradiant modes are reflected from infinity,
- This cause the effect of black hole bomb in Kerr metric with reflecting envelope (or asymptitically AdS blak holes). This instability may be also interpreted as scalarization (creation of scalar clouds arond black holes)

Constrainted Benenti-Francaviglia ansatz and integrability

- Our goal here is to obtain general EMDA solution with LD asymptotic using recently proposed method [1,2] of direct integration of EMDA equations for stationary axisymmetric Carter-Benenti-Francavigilia metrics

$$ds^2 = \frac{A_2}{\Sigma}(bdt - Bd\varphi)^2 - \frac{B_2}{\Sigma}(adt - Ad\varphi)^2 - \frac{\Sigma}{A_2}dr^2 - \frac{\Sigma}{B_2}dy^2,$$

  where one-variable functions $A = A(r),\ A_i = A_i(r),\ B = B(y),\ B_i = B_i(y)$ are introduced, and $a,\ b$ are constants. BF metrics ensure existence of slice-reducible Killing tensor, ensuring separability of Hamilton-Jacoby equations and existence of two null shearfree geodesics congruences while the metric is non-algebraically special one.
- In addition, the following constraint is imposed $\Sigma = \sqrt{-g} = A - aB$, which guarantees the separation of the variables in the Klein-Gordon equation for the scalar field.
- Under such conditions, the EMDA equations can be directly integrated, even in presence of the potential, depending on dilaton and axion (gauged supergravity) [2]

- Introduce complex axidilaton by $z = \kappa + ie^{-2\phi}$ and rewrite the action as

$$S = -\frac{1}{16\pi}\int\left(R + \frac{2\nabla z\nabla\bar{z}}{(z-\bar{z})^2} - (iz\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + c.c.)\right)\sqrt{-g}d^4x,$$

where the self-dual Maxwell tensor is $\mathcal{F}^{\mu\nu} = \frac{1}{2}\left(F^{\mu\nu} + i\tilde{F}^{\mu\nu}\right)$. The equation of motion for the complex axidilaton than takes the form

$$\Box z - \frac{2\partial z\partial z}{(z-\bar{z})} - \frac{(z-\bar{z})^2}{4}\left(iF_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\tilde{F}^{\mu\nu}\right) = 0.$$

- It is straightforward to check that the equations of motion(but not the action) are invariant under the S-duality transformations non-changing the metric:

$$z \to \frac{az+b}{cz+d}, \quad \mathcal{F}^{\mu\nu} \to (c\bar{z}+d)\mathcal{F}^{\mu\nu}, \quad ad - bc = 1.$$

Tetrad representation

- Introduce the set of tetrad one-forms e^a ($a = 1, .., 4$), associated with our metric parametrisation

$$e^1 = \alpha(bdt - Bd\varphi),$$
$$e^2 = \beta(adt - Ad\varphi),$$
$$e^3 = \alpha^{-1}dr,$$
$$e^4 = \beta^{-1}dy,$$

where

$$\alpha = \sqrt{A_2/\Sigma}, \qquad \beta = \sqrt{B_2/\Sigma}$$

.

- Than the line element will be given by

$$ds^2 = \eta_{ab}e^a e^b, \qquad \eta_{ab} = diag(1, -1, -1, -1).$$

Einstein tetrad equation

- Tetrad components of the Einstein equations can be presented using the Ricci tensor as

$$R_{ab} = T^{sc}_{ab} + \frac{(z - \bar{z})}{2i} T^{em}_{ab},$$

where the reduced scalar term without trace part is equal to

$$T^{sc}_{ab} = -\frac{1}{(z - \bar{z})^2}(z_{,a}\bar{z}_{,b} + z_{,b}\bar{z}_{a}),$$

while the "pure" Maxwell term is

$$T^{em}_{ab} = 2\left(F_{ac}F^{c}_{b} + \frac{\eta_{ab}}{4}F_{cd}F^{cd}\right),$$

- Tetrad components of the modified Maxwell equations can be written as

$$F^{ab}\partial_a(z - \bar{z}) + (\nabla_\mu F^{\mu\nu})e^b_\nu(z - \bar{z}) + i\tilde{F}^{ab}\partial_a(z + \bar{z}) = 0.$$

Maxwell equations

- An ansatz for the vector one-form compatible with our metric was found by Carter

$$A_{[1]} = \frac{R}{\alpha\Sigma} e^1 + \frac{Y}{\beta\Sigma} e^2,$$

which leads to the following nonvanishing tetrad components of the Maxwell tensor $F = dA$:

$$F_{13} = -\tilde{F}_{24} = \frac{A'(R + aY) - \Sigma R'}{\Sigma^2}, \quad F_{24} = \tilde{F}_{13} = -\frac{B'(R + aY) + \Sigma Y'}{\Sigma^2}.$$

- The only nonvanishing components of the modified Maxwell equations are

$$F_{13}(z - \bar{z})_{,r} + \frac{(z - \bar{z})}{\Sigma} \left[A' F_{13} + B' F_{24} - \Sigma F_{13,r} \right] + iF_{24}(z + \bar{z})_{,r} = 0,$$

$$F_{24}(z - \bar{z})_{,y} - \frac{(z - \bar{z})}{\Sigma} \left[a(A' F_{13} + B' F_{24}) - \Sigma F_{24,y} \right] - iF_{13}(z + \bar{z})_{,y} = 0,$$

while the Bianchi identities are satisfied by default.

Maxwell field

- From our parametrisation we obtain that the Maxwell energy-momentum tensor can be presented as follows:

$$T_{11}^{em} = T_{22}^{em} = -T_{33}^{em} = T_{44}^{em} = F_{13}^2 + F_{24}^2$$

- One can introduce the tetradic potential for non-zero part of Maxwell tensor using

$$F_{13} = -\frac{\partial}{\partial r}\left(\frac{R + aY}{\Sigma}\right), \quad F_{24} = -\frac{1}{a}\frac{\partial}{\partial y}\left(\frac{R + aY}{\Sigma}\right).$$

- From Maxwell equations one can derive a simple linear equation on functions R, Y :

$$aR'' - Y'' = 0.$$

- It follows from here that R, Y are at most quadratic polynomials of respective variables:

$$R = R_0 + R_1 r + cr^2, \qquad Y = Y_0 + Y_1 y + c(ay^2),$$

where R_i, Y_i are real constants and c is the separation constant.

Axidilaton equations

- The R_{34} component of the Ricci tensor is identically zero, so Einstein equations leads to first separate equation for axidilaton

$$z_{,r}\bar{z}_{,y} + z_{,y}\bar{z}_{,r} = 0.$$

- Also hold the following identities

$$R_{11} + R_{33} = \alpha^2 \mathcal{R},$$
$$R_{22} - R_{44} = -a^2\beta^2 \mathcal{R},$$
$$\mathcal{R} = \frac{1}{2\Sigma^2}\left[(A')^2 + (B')^2 - \Sigma A''\right],$$

from which one obtains the second separate axidilaton equation:

$$a^2 z_{,r}\bar{z}_{,r} - z_{,y}\bar{z}_{,y} = 0.$$

- From these two equations it follows that z is a holomorphic or antiholomorphic function of $w = r + iay$, except at the poles. So we can take

$$z = f(w), \quad w = r + iay.$$

Axidilaton anzats

- Then $z_{,y} = -iaz_{,r}$, and the Laplace equation holds
$$z_{,yy} = -a^2 z_{,rr},$$
so we deal with a harmonic function.
- A holomorphic function is needed, one that is single-valued in the entire complex plane and non-singular except at a simple pole. Such a function must be a fractional-linear transformation
$$z = \frac{c_1(r+iay)+c_2}{c_3(r+iay)+c_4},$$
where all c_i are arbitrary complex constants, and the constraint $c_1c_4 - c_2c_3 \neq 0$ must be satisfied for invertibility.
- As shown in [2], generic case correponds to AF or asymptotically ADS black holes
- Degenerate case $c_1 = 0$ turns out to produce ALD black holes

Polynomial structure of BF functions

- From the R_{12} component of the Einstein equations which has no the source term we have:

$$aA'' + bB'' = 0.$$

Since the first term depends only on r, and the second only on y, it follows that the functions A, B must be polynomials of the second degree of the corresponding variables.

- Thus, the most general expression for these quantities is

$$A(r) = \alpha_0 + qr + gr^2,$$
$$B(y) = C + ny - agy^2,$$

where α_0, C, q, n, g are arbitrary constants and where we have set $b = 1$ by rescaling of the time coordinate t.

- Next the strategy consists in extracting from tetrad Einstein equations further linear equations on BF coefficients. The first is the linear equation following from the difference of $R_{11} - R_{22}$ components:

$$A_2'' + B_2'' = 0,$$

- This is solved as

$$A_2 = a_0 - 2mr + \lambda r^2, \qquad B_2 = b_0 + 2b_1 y - \lambda y^2,$$

where $a_0, m, b_0, b_1, \lambda$ are real constants. By rescaling of the coordinate φ one can set $\lambda = 1$, so for the functions A_2, B_2 we have

$$A_2 = a_0 - 2mr + r^2, \qquad B_2 = b_0 + 2b_1 y - y^2,$$

- In order to obtain the solution with linear dilaton asymptotics for the axidilaton we take the degenerate form of the fractional-linear function with $c_1 = 1$, than

$$z = \frac{c_2}{c_3(r + iay) + c_4}.$$

- We use the invariance of the metric parametrization under a constant shifts of the coordinates r and y for the following simplifications. First we use the shift $r \to r + r_0$ to remove the real part of the complex constant c_4, and than use $y \to y + y_0$ to remove the linear term in the function B_2, and after that combine constants into one complex constant $d = c_2/c_3$. Finally for the axidilaton we will have the following simple expression

$$z = \frac{d}{r + iay - ip}.$$

- Than by no loosing generality we can choose the function B_2 in the form:

$$B_2 = 1 - y^2,$$

so the variable y can be further identified with the cosine of the azimuthal angle $y = \cos\theta$.

General ALD solution

Substituting all our functions into the remainig equations of motion we obtain that the variables are separated, and than the most general solution with second rank Killing tensor and linear dilaton asymptotics is given by the functions:

$$A_2 = r^2 - 2mr + a^2 - p^2 + \frac{(mn - qp)^2}{n^2 + q^2},$$

$$B_2 = 1 - y^2,$$

$$A = qr + pn + aC,$$

$$B = ny + C,$$

$$R = \frac{r^2}{\sqrt{2d_0}} + \frac{mp}{\sqrt{2d_0}}\frac{n}{q} - a\delta,$$

$$Y = \frac{ay^2}{\sqrt{2d_0}} - \frac{y}{\sqrt{2d_0}}\left(p + m\frac{n}{q}\right) + \delta,$$

$$z = id_0\frac{(q - in)}{r + iay - ip}.$$

Interpretation of parameters: conical singularity

To clarify the meaning of extra parameters, it is useful to write the metric in Kaluza-Klein form:

$$ds^2 = F(dt - \omega d\varphi)^2 - \frac{dl_3^2}{F},$$

where dl_3^2 is the 3-metric spanned by r, θ, φ. Here we will start by providing the search of the conical singularity in our solution. Consider the part of the spacetime metric spanned by the coordinates r and φ:

$$dl_{(r,\varphi)}^2 = \frac{\Sigma}{A_2}\left[dr^2 + \frac{A_2}{\Sigma^2}\left(A^2 B_2 - B^2 A_2\right)d\varphi^2\right].$$

For the linear dilaton solution the coefficient Σ/A_2 diverges as $r \to \infty$, because Σ is just a linear function of radial coordinate while A_2 if quadratic. So, we will consider conformally connected metric $d\tilde{l}^2 = (A_2/\Sigma)dl^2$, and if the original metric had a conical singularity than it is also true for the conformal metric as the conformal transformation doesn't affect the angles.

Misner string

Than, restricting our consideration to the equatorial plane, where we'll have $B_2 = 1, \; B = C$ and than taking the limit $r \to \infty$ one obtains

$$d\tilde{l}^2_{(r,\varphi)} = dr^2 + r^2 \left[1 - \left(\frac{C}{q} \right)^2 \right] d\varphi^2,$$

and we see that in general there is actually a cosmic string in the solution, which can be removed by imposing $C = 0$.
Consider now the rotation function

$$\omega = -\frac{g_{t\varphi}}{g_{tt}} = \frac{A_2 B - a B_2 A}{A_2 - a^2 B_2},$$

which in the limit $r \to \infty$ is significantly simplified to

$$\omega = B = ny + C,$$

so it is actually the parameter C alone, which is responsible for changing the configuration of the Misner strings and also for the presence of the conical singularity. In order to set the Misner strings symmetric and also exclude the conical singularity from the solution we demand $C = 0$.

S-duality scale and fluxes

One check that the parameter d_0 is nothing else but the parameter of the S-duality transformation corresponding to the rescailing of the axidilaton and the Maxwell field by

$$z \to d_0 z, \quad \mathcal{F}^{\mu\nu} \to \frac{1}{\sqrt{d_0}} \mathcal{F}^{\mu\nu},$$

so, without loosing the generality we can set $d_0 = 1$.
To find the electromagnetic charges one can calculate the fluxes of electric and magnetic fields through the sphere at infinity

$$Q = \frac{1}{4\pi} \oint_{S_r} E^r \sqrt{-g}\, d\Omega, \qquad P = \frac{1}{4\pi} \oint_{S_r} H^r \sqrt{-g}\, d\Omega.$$

Radial components of these fields read

$$E^r = e^{-2\phi} F^{rt} + \kappa \tilde{F}^{rt} = -\frac{A_{23}}{2\Sigma} \Big(i(z - \bar{z}) F_{13} - (z + \bar{z}) F_{24} \Big),$$

$$H^r = \tilde{F}^{rt} = -\frac{A_{23}}{\Sigma} F_{24}.$$

Fluxes at infinity

Substituting the solution and taking $r \to \infty$ one can obtain for electric charge

$$Q = \frac{1}{\sqrt{2}q}(q^2 + n^2),$$

while the magnetic charge diverges linearly as

$$P = -\frac{1}{\sqrt{2}}\frac{n}{q}r + \frac{p(q^2 + n^2) + mnq}{\sqrt{2}q^2} + O\left(\frac{1}{r}\right).$$

So, the parameter n, which plays a role of a NUT parameter leads to the infinite flux of the magnetic field. Than, setting this parameter to zero one is left with the following charges

$$Q = \frac{q}{\sqrt{2}}, \qquad P = \frac{p}{\sqrt{2}},$$

so, the parameter q plays a role of the electric charge, while the parameter p should be identified with the magnetic charge of the black hole. Note that Q is associated with the flux supporting the LD background, and not with the black hole on it. In contrary, P must be attributed to magnetic charge of the black hole

Rod structure

- If $n \neq 0$ (non-zero NUT parameter) the solution acquires non-trivial behavior at the polar axis, known as rod structure (Harmark, 2004). This leads to mass formulas prescriptions in EMDA gravity in terms of individual parametrs of mass, anqualr momentum and electric charge to each rod (see **I.Bogush, G.Clément and D.Gal'tsov, "Mass formulas for supergravity black holes with string singularities," Eur. Phys. J. C 84, no.7, 727 (2024) [arXiv:2405.19196 [gr-qc]]** and refs. therein)
- Each rod is a segment of the axis at which the two-dimensional eigenvector of the Gram matrix is constant, its components being the angular momentum and the surface gravity of Killing horizons (spacelike on Misner strings). The sum of conserved quantities calculated over the cylinders around the rods by Ostrogradski theorem must be equal to the corresponding quantities calculated on the spheres at infinity
- Magnetic charge does not contribute to the black hole horizon rod mass, but contributes to asymptotic mass
- For ALD black holes subtraction for mass is needed.

We use the above technique to reveal the divergent part of the flux in the case of nonvanishing NUT. For regularization, consider cylinders around the finite segments of Misner strings:

- z_+ northern string $r \in [r_H, l), \;\; \cos\theta = 1;$
- z_H the horizon rod $r = r_H, \;\; \theta \in [0, \pi];$
- z_- southern string $r \in [r_H, l), \;\; \cos\theta = -1;$

Where l is the length of the cylinder. Calculating now the magnetic fluxes around around the rods one obtains for the sum $P = P_H + P_+ + P_-$

$$P = -\frac{(ql + np)[n(l^2 + a^2 - p^2) - l(pq + mn)]}{\sqrt{2}[(ql + np)^2 - a^2 n^2]},$$

which exactly coincides with the flux through the sphere of some fixed radius r as long as $r = l$. Thus, the growing flux of the magnetic field at spatial infinity is caused by the Misner strings.

Conclusions

- ALD solutions and their role in supergravity seems underexplored. They create a number of new features such as scalarization, new holography, new photon structure and so on
- General solution of EMDA theory describing ALD black holes and its nutty generalizations is shown to correspond to degenerate axidilaton structure within the recently constructed integrable EMDA model
- General solution obtained contains several new parameters including magnetic charge, shift parameters for Misner and Dirac string, scale parameter and conical parameter
- One novel feature with respect to AF and ADS black holes is that Misner string shift parameter is related to conical parameter, so the solution without conical singularity extsts only for symmetric configuration of Misner strings
- An interesting interplay between magnetic fluxes at infinity and though the Misner strings is manifest exhibiting conservation of diverging parts
- More details will be given in preprint under preparation