Stochastic gravitational wave (GW) background from hyperbolic encounters

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Timeliness of the topic

- Gravitational Wave (GW) astronomy promises to observe different kinds of astrophysical sources.
- Remarkable achievements in observations! Compact binary coalescence (CBC).
- Most binaries are expected to circularize before entering LIGO band (Peters & Mathews 1963)
- What about the dynamical captures? eccentric or scattering/hyperbolic orbits? (Vittori et. al. 2012, Nagar et. al. 2021) Topic of the present talk..
- We need the exact waveform, and modified search algorithm (Morras et. al. 2022, Albanesi et. Al. 2025) --- a long way to go!

References

- Detectability of stochastic gravitational wave background from weakly hyperbolic encounters, A & A (2024)
- Gravitational wave observatories may be able to detect hyperbolic encounters of black holes, MNRAS (2021)

List of collaborators

- Morteza Kerachian, ASU, Prague
- Georgios Lukes-Gerakopoulos, ASU, Prague
- Sanjit Mitra, IUCAA, Pune
- Sourav Chatterjee, TIFR, Mumbai

Hyperbolic encounters: a brief history from the past

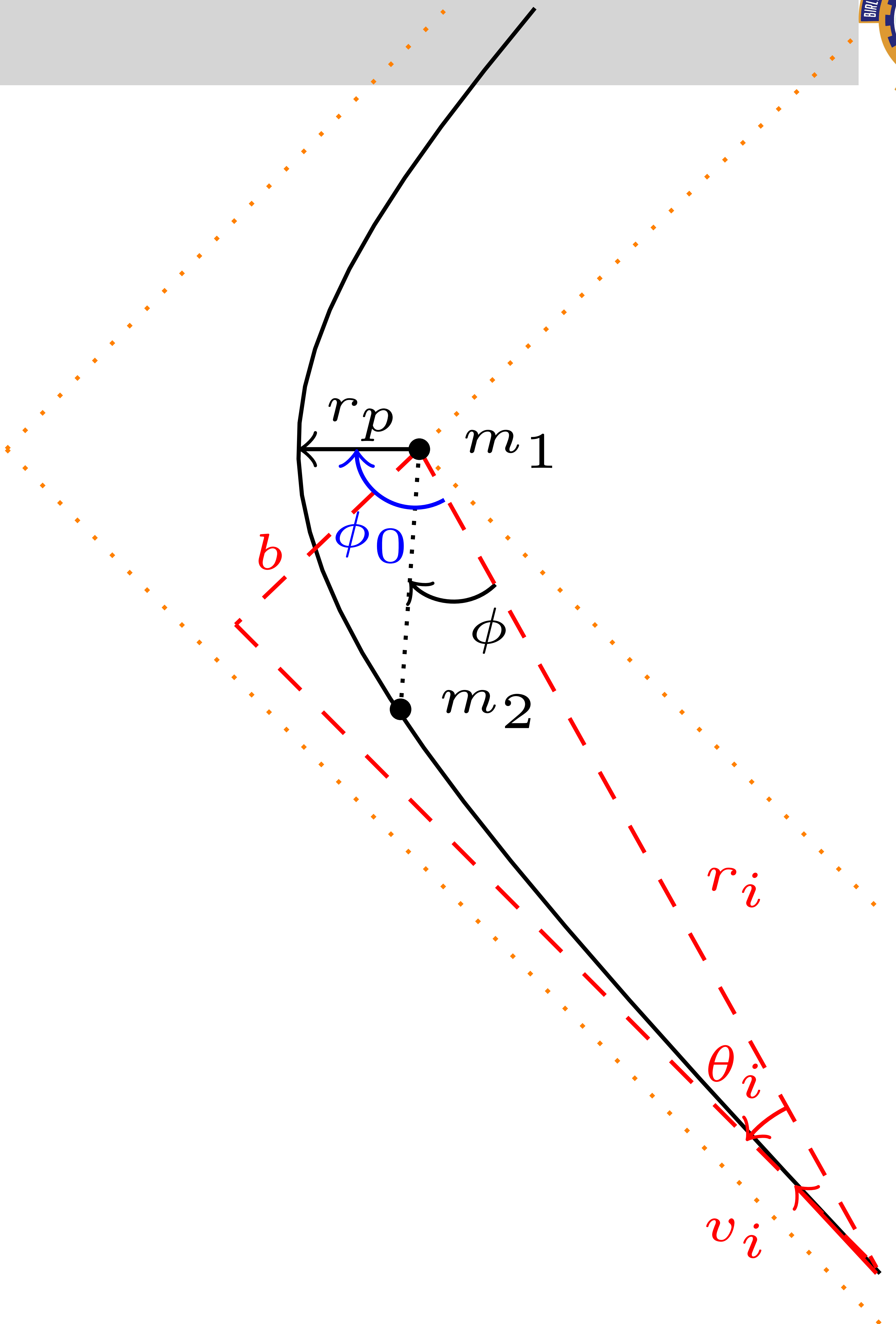
- Hyperbolic interactions and their implications in gravitational wave (GW) astronomy are relatively new, and explored extensively in recent times.
- Hyperbolic encounters in dense clusters are relevant
- These events are burst-like and transient
- We would need modified search algorithm and waveform modeling to detect these signals
- How about the stochastic background from these encounters? We will discuss on this aspect..

Part - I

The model of interaction

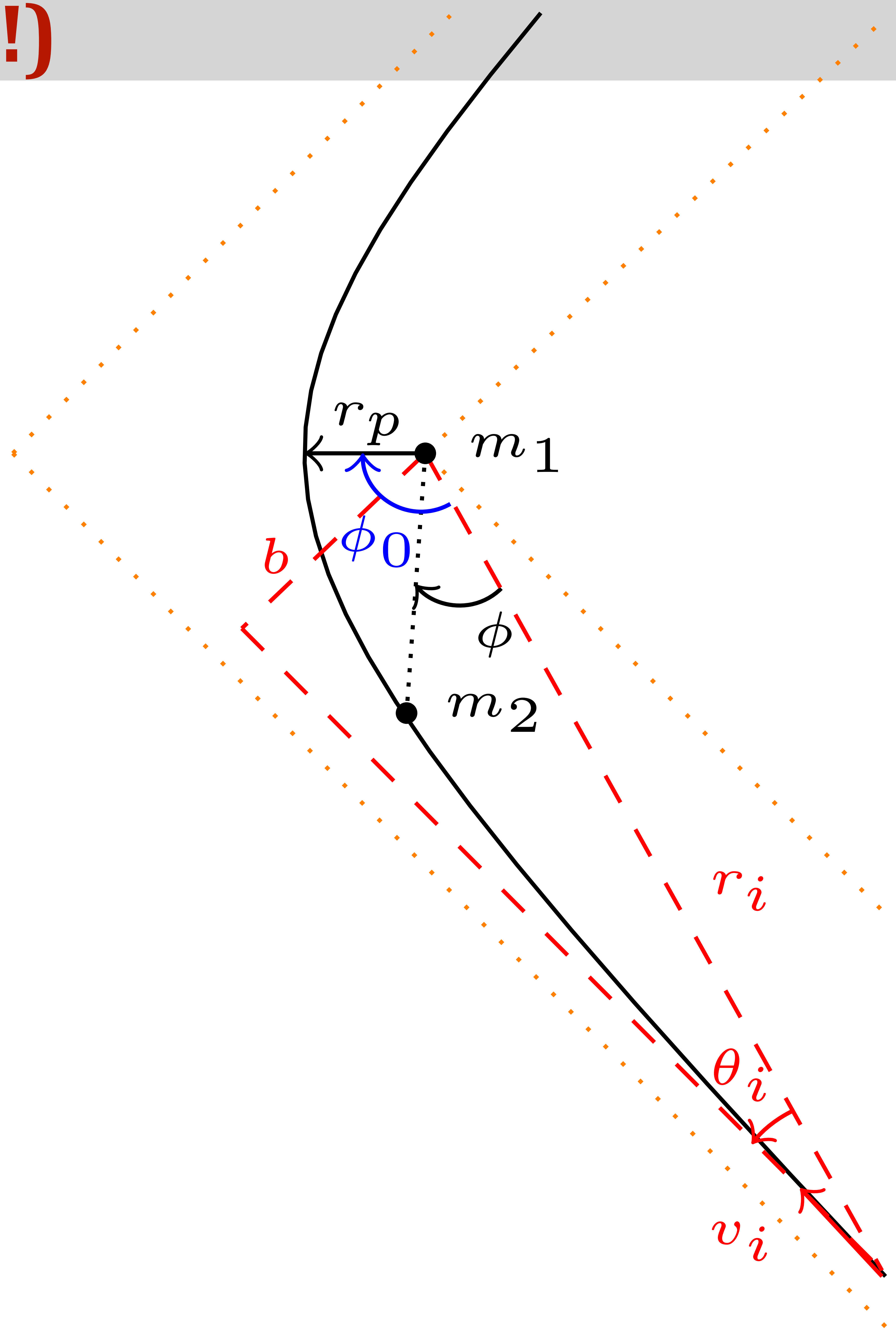
Trajectory: orbital set up

- Initial position and angle: r_i & θ_i
- Initial velocity: v_i
- Masses of the binary: m_1 & m_2
- Impact parameter: b
- The eccentricity: e
- Periapsis/closest distance: r_p
- Angle at the periastron: ϕ_0
- Coordinate: ϕ



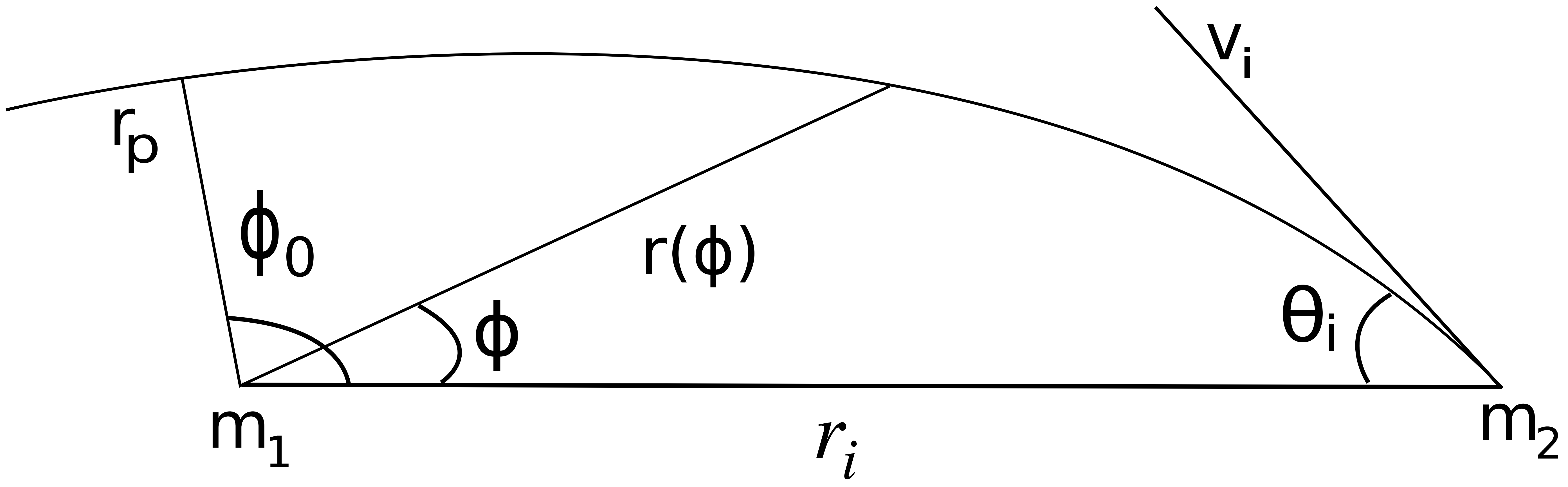
Trajectory: orbital set up (a correction!)

- We are interested to study hyperbolic encounters inside a closed cluster of finite radius (R_c)
- Typically, in a hyperbolic event, the initial distance between the binary constituents is infinite!
- However, if we assume the cluster is closed, the initial distance r_i can be at max $\sim R_c$, not infinite!
- Similarly θ_i, the initial angle between the binary constituents, can have a wide range!



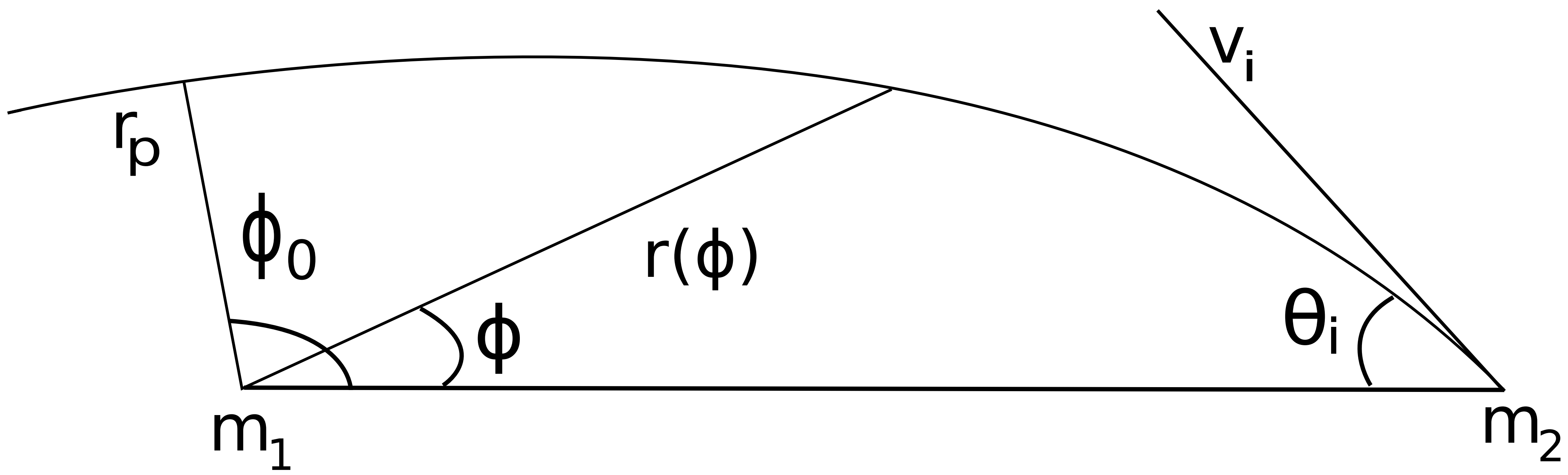
Hyperbolic interaction inside a closed cluster: a closer look

- Initial position and angle: r_i & θ_i
- Initial velocity: v_i (virial velocity)
- Masses of the binary: m_1 & m_2
- Periapsis/closest distance: r_p
- Position: $r(\phi)$
- Coordinate: ϕ
- Angle at the periastron: ϕ_0



Hyperbolic interaction inside a closed cluster: a closer look

- Conservation of momentum and energy are assumed
- Earlier works are based on scattering problem assuming initial distance to be infinity
- Need to redefine the initial conditions based on local parameters — initial distance (r_i), initial angle (θ_i), and v_i, which we assumed as ***virial velocity***
- While $r_i \leq R_c$, the cluster's radius taken as 10pc, we assume $\theta \in (\theta_{\min}, \theta_{\max})$. Decide $\theta_{\max}$ from threshold signal to noise ratio (SNR), and $\theta_{\min}$ from Schwarzschild radius(r_s)



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- Initial velocity: v_i (virial velocity)
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The orbital parameters: new terms

The eccentricity e, angle at periapsis ϕ_0, and periapsis distance r_p

$$e^2 = 1 + \frac{L^2 vi^2}{G^2M^2}\left\{1 - \left(\frac{2GM}{v_i^2}\right)\frac{1}{r_i}\right\},$$

$$\tan\phi_0 = \frac{Lv_i\cos\theta_i}{L^2/r_i - GM} = -\frac{Lv_i}{GM}\left\{\frac{\cos\theta_i}{1 - L^2/(GMr_i)}\right\},$$

$$r_p = \frac{L^2\cos\phi_0}{L^2/r_i - GM(1-\cos\phi_0)} = \frac{L^2}{GM(1+e)}$$

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With

$r_i \to \infty$, and $\theta_i \to 0$,

Momentum $L = r_i v_i \sin\theta_i$ remains finite, and we obtain

$$e^2 = 1 + \frac{L^2 vi^2}{G^2M^2}, \tan\phi_0 = -\frac{Lv_i}{GM}, r_p = \frac{L^2}{GM(1+e)}$$

The orbital evolution in time

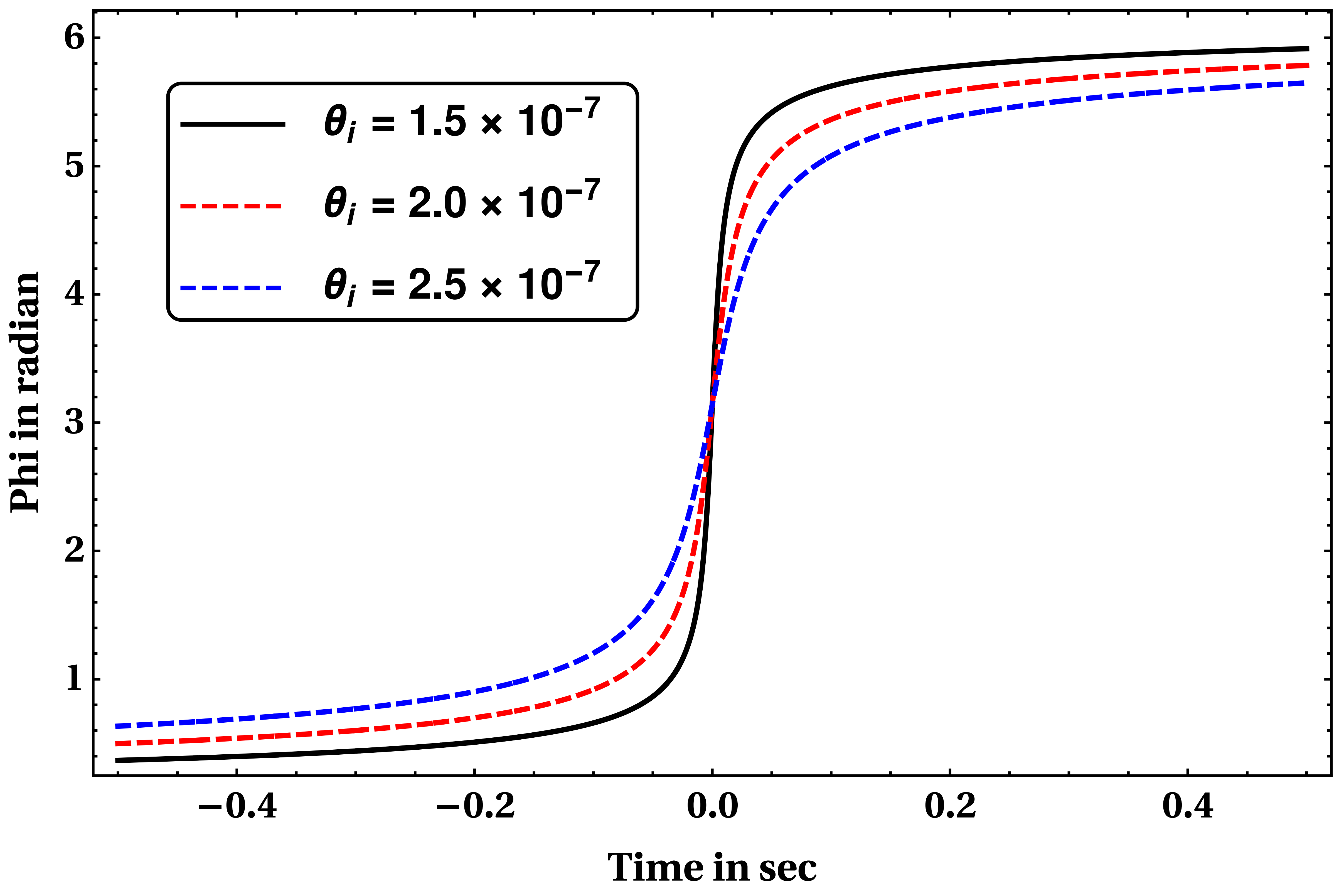
- The orbital trajectory can be characterized by

$$r(\phi) = \frac{L^2/(GM)}{1 + \left[L^2/(GMr_p) - 1\right] \cos(\phi - \phi_0)}$$

which follows the initial condition, at $\phi = \phi_0, r = r_p$.

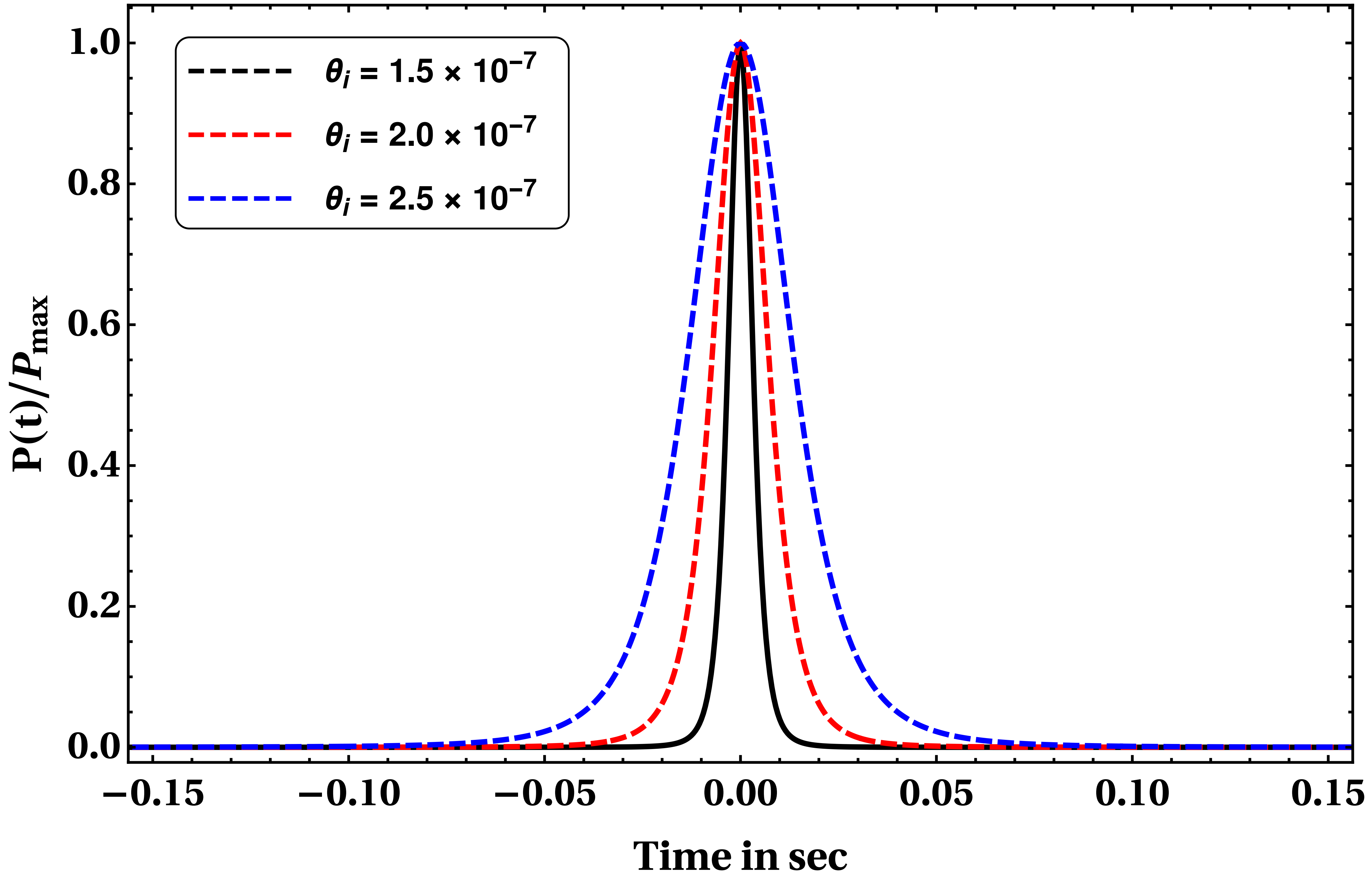
The orbital evolution in time

- Figure represents the $\phi \ vs \ t$ plot
- We have $r_i = 1$pc, and $m_1 = m_2 = 10M_{\odot}, v_i \cong 10.69$km/sec
- Note $L = mr(\phi)^2\dot{\phi}$, where "dot" is a differentiation with respect to time
- The $t = 0$ represents the closest distance, $r = r_p$
- Lesser the θ_i, stronger the interaction!

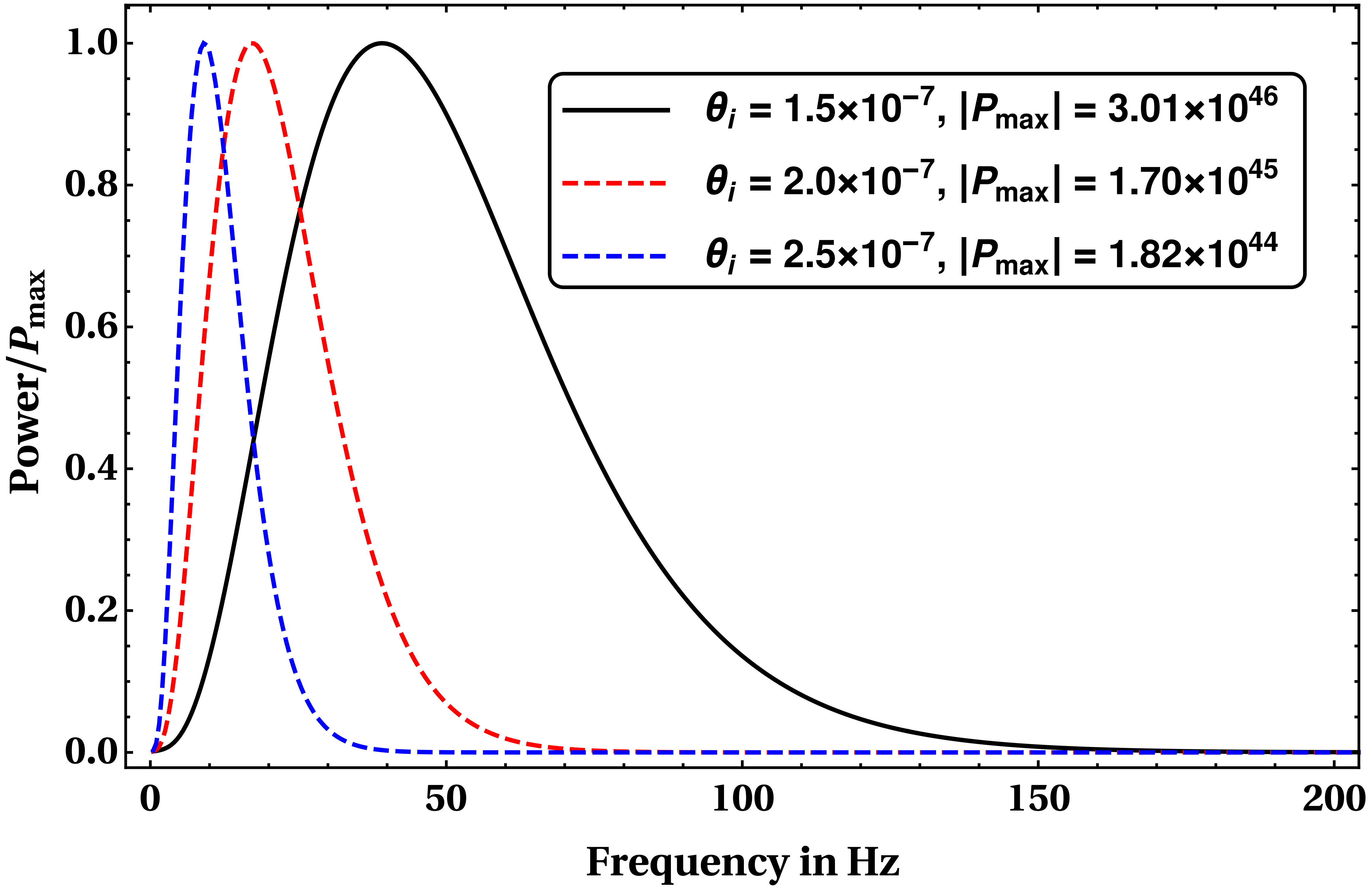


The power radiation in time and frequency domain

Time Domain



Frequency Domain



Normalized power radiation with $r_i = 1\text{pc}, m_1 = m_2 = 10M_\odot$

Quick summary

- We revised our model of interaction.
- The orbital dynamics explicitly depends on the initial conditions, namely initial distance and angle.
- The power peaks at the periapsis, the closest distance that can be approached.
- The obtained the power spectrum is consistent with existing literature!

Part - II

The stochastic background

Stochastic gravitational wave background (SGWB) from an event

- The dimensionless GW energy density spectrum is given as (Bellido et. al. 2022, Maggiore 2018)

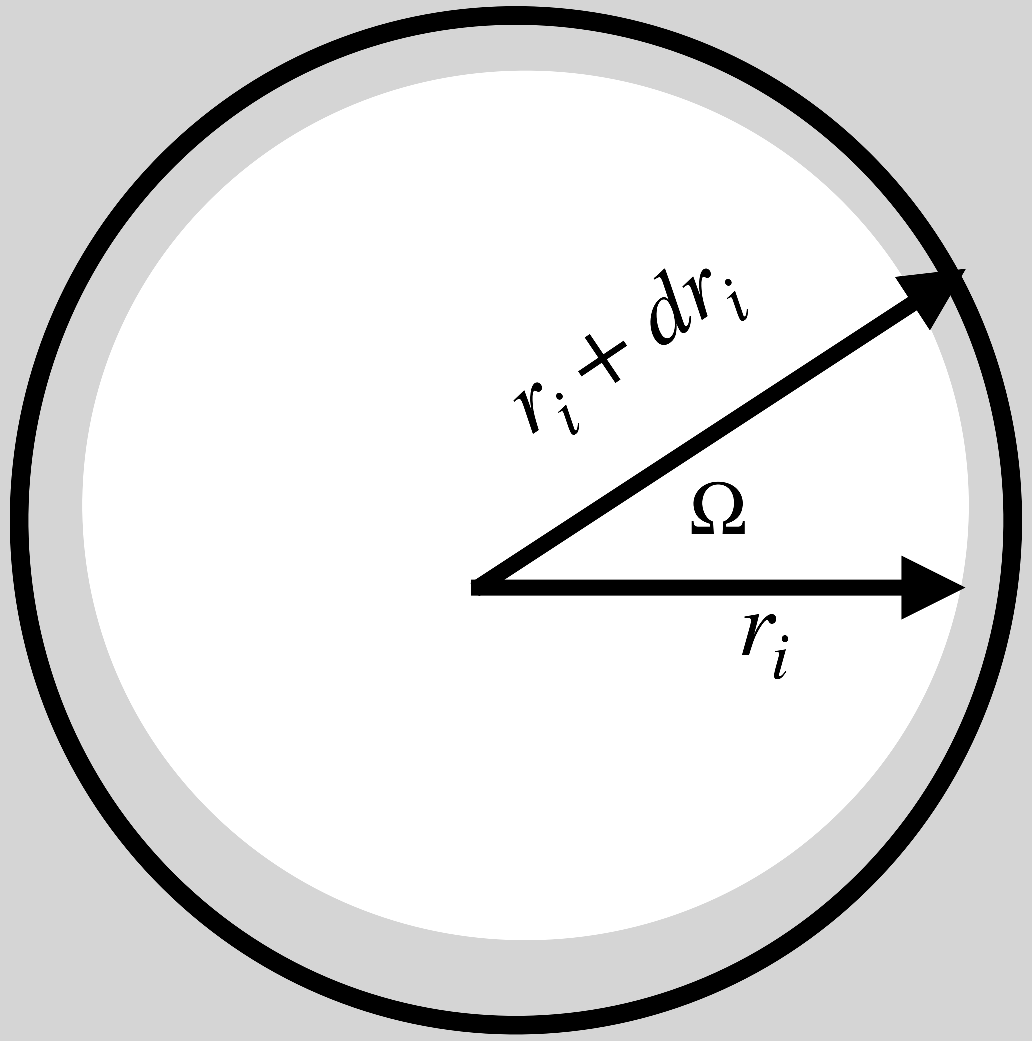
$$\Omega_{\rm GW}(f) = \frac{1}{\rho_c}\int_0^\infty \frac{dz}{1+z}N(z)\frac{dE_{\rm GW}(f_r)}{d\ln f_r}$$

Here $\rho_c = 3H_0^2/(8\pi G)$, $N(z)$ is the density of GW events at redshift z, and f_r is the redshifted frequency. Other quantities have usual meanings.

- To obtain the above expression, we need to find the density of event rates for hyperbolic events inside a cluster.

Single hyperbolic event (inside a closed cluster)

- Given a fixed r_i, the solid angle Ω as a function of $\theta_{\max}(r_i)$ and $\theta_{\min}(r_i)$, which results in a detectable signal $(\theta_i \ll 1)$



$$\Omega = 2\pi \int_{\theta_{\min}(r_i)}^{\theta_{\max}(r_i)} \sin\theta d\theta = \pi\left[\theta_{\max}(r_i)^2 - \theta_{\min}(r_i)^2\right]$$

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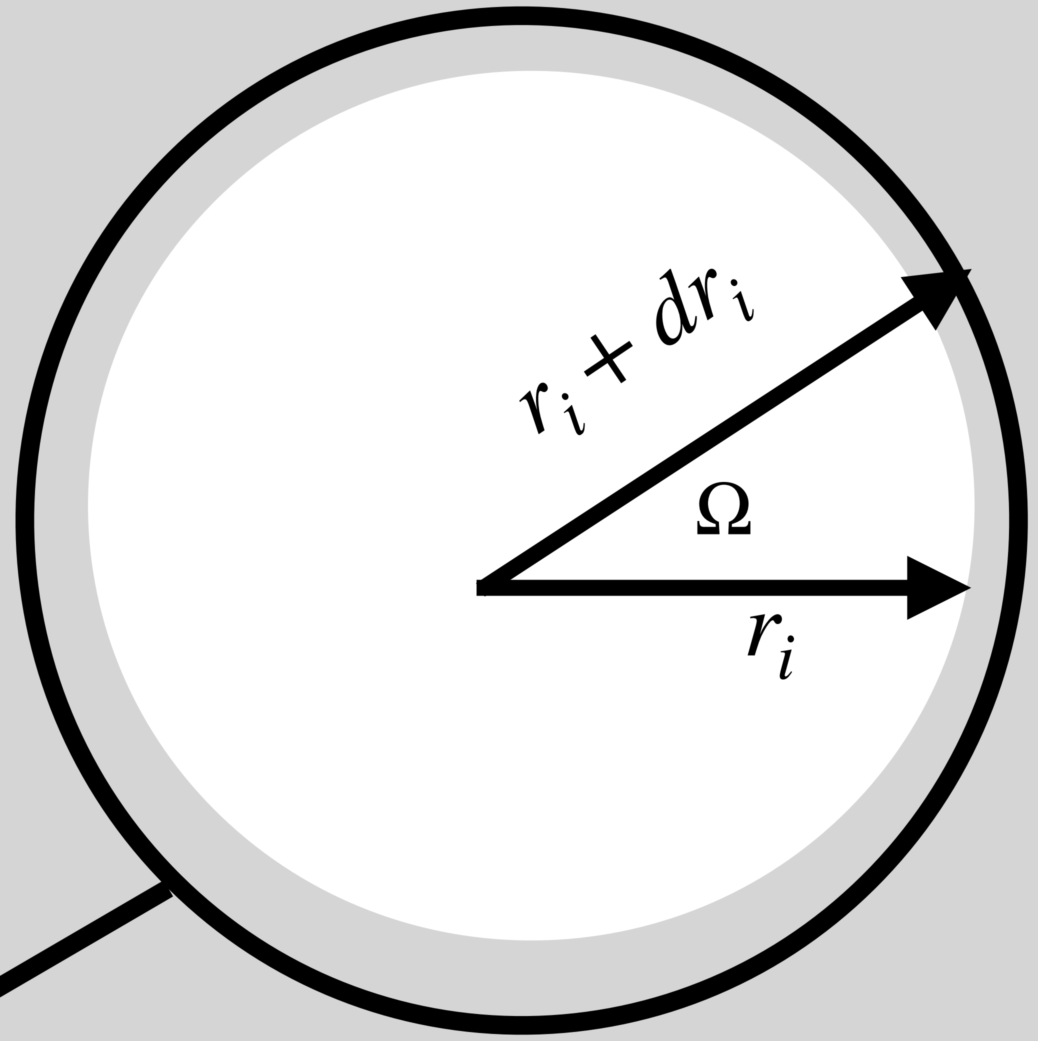

Note that $\theta_{\max}$ is obtained from the SNR constrained (detector dependent), and $\theta_{\min}$ from the condition $r_p \geq 2r_s$, where r_s is the Schwarzschild radius (orbital model dependent).

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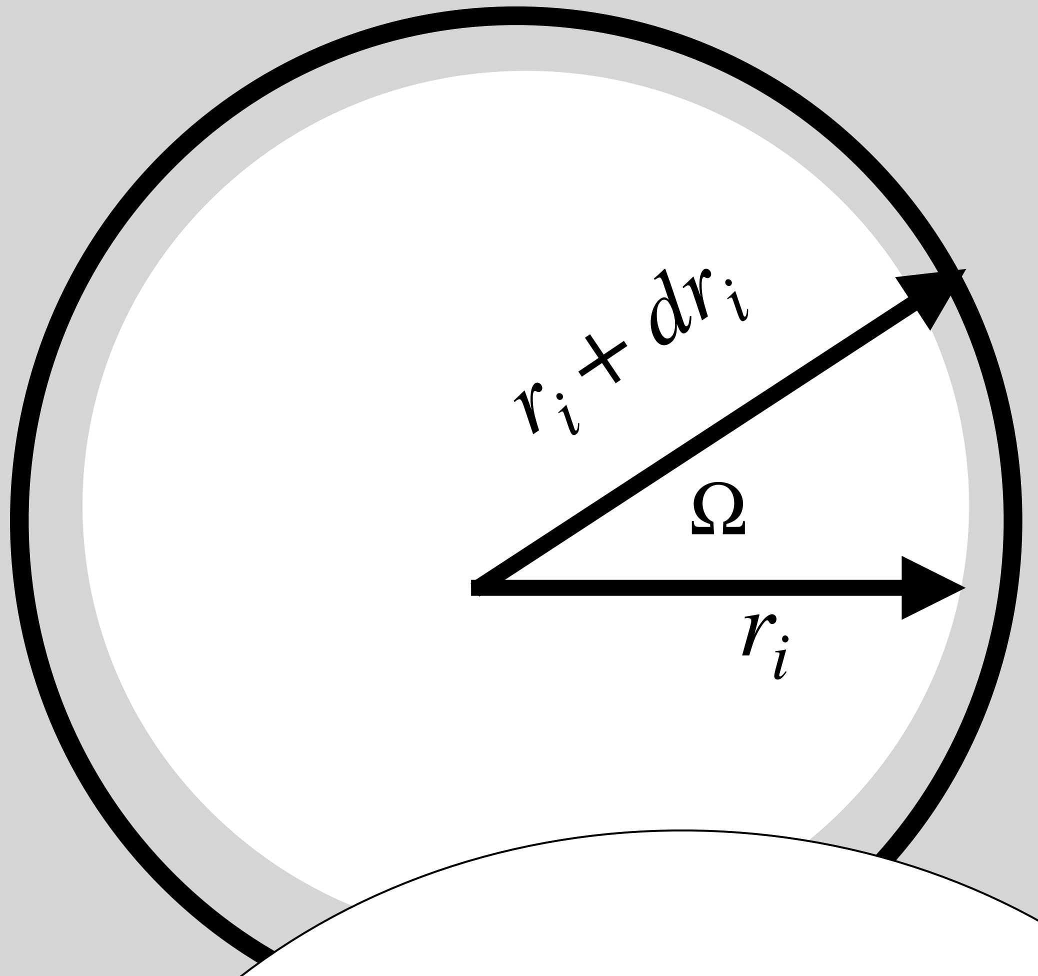

- Assuming that the objects inside the cluster are uniformly distributed, the probability of selecting a fraction of particles fall within the above solid angle is

$$P_{\text{theta}} = \frac{\Omega}{4\pi} = \frac{1}{4}\left[\theta_{\max}(r_i)^2 - \theta_{\min}(r_i)^2\right]$$

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- Therefore, for an individual compact object inside the cluster, number of events per unit time is given as

$$P_{\text{indv}} = \int_{R_{\min}}^{R_{\text{c}}} \left(\frac{P_{\text{theta}}}{t_{\text{col}}}\right) 4\pi r_i^2 n_{\text{s}} dr_i$$

- In this, t_{col} is the collision time, and $t_{\text{col}} \leq r_i/v_i$, $n_{\text{s}} = 3n_{\text{co}}/(4\pi R_{\text{c}}^3)$ is number density of compact objects inside the cluster, and $R_{\min}$ is the lower radial cutoff. Note n_{co} is the number of compact objects which is redshift dependent!

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- Hence by combining r_i & θ_i both, we get

$$P_{\text{indv}} = \int_{r_i} \int_{\theta_i} \frac{1}{4\pi} (2\pi \sin\theta_i) d\theta_i (4\pi r_i^2 n_s) dr_i,$$

$$= \int_{r_i} \int_{\theta_i} 2\pi n_s r_i^2 \sin\theta_i d\theta_i dr_i \, .$$

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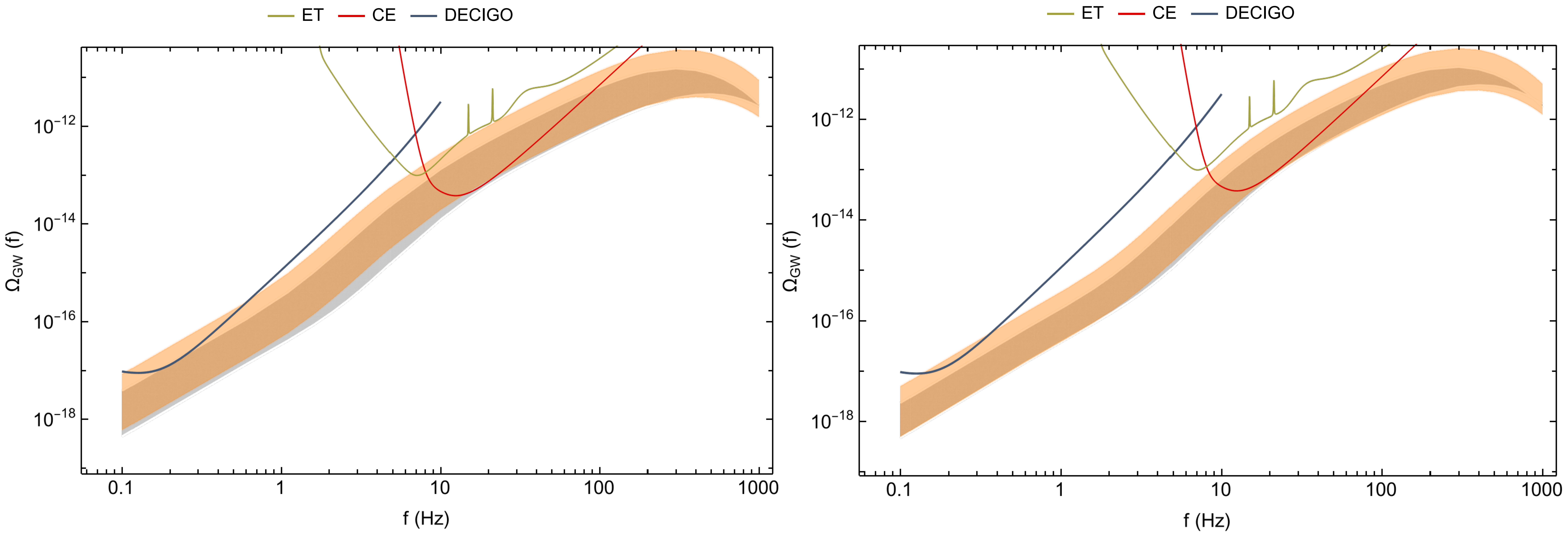
- By ignoring small scale structure, we have $P_{\rm cluster} = n_{\rm co} P_{\rm indv}$.

SGWB from hyperbolic events



Background from hyperbolic encounters for different cluster properties and binary parameters

SGWB from hyperbolic events

- Total contribution to the SGWB from weakly hyperbolic encounters ($e \sim 1 + \epsilon$, where ϵ is a small number).

- In this plot, we consider

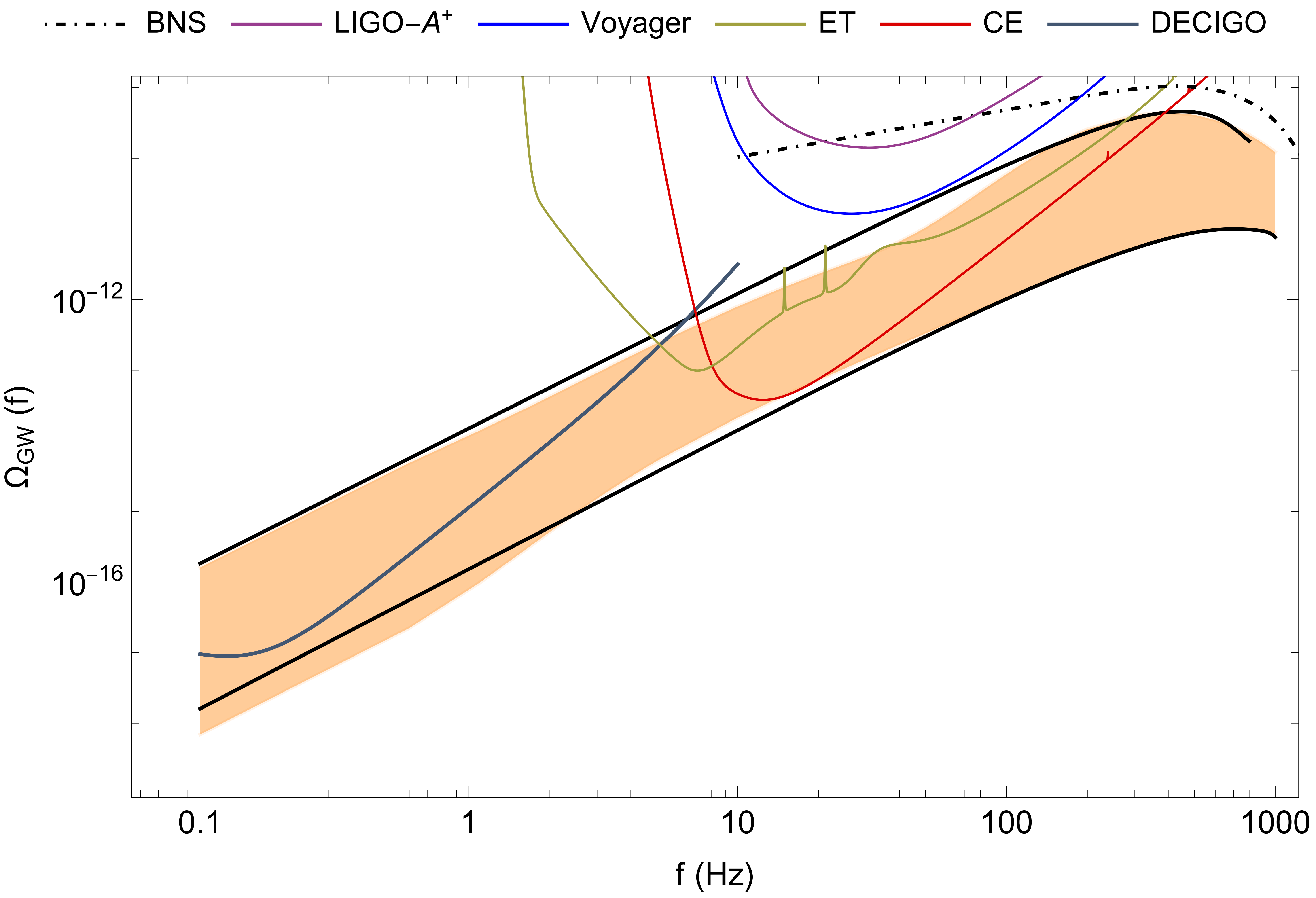
$$5\ km/s \leq v_i \leq\ 18 km/s$$

$$\{r_{imin}, r_{imax}\} = \{0.03\text{pc}, 9.03\text{pc}\}$$

$$15M_\odot \lesssim m_1 \leq 50M_\odot$$

$$5M_\odot \lesssim m_2 \leq 25M_\odot$$

- The bottom line represents the pessimistic scenario, while the top line is the optimistic scenario



Discussions

- In this work, we studied the SGWB from hyperbolic encounters inside bound compact clusters. Specifically, we investigate weakly hyperbolic encounters with eccentricities close to one, and compute the energy density of the SGWB.
- As discussed, the SGWB from these encounters falls within the range of third generation GW detectors, primarily Cosmic explorer.
- Given we accumulate more encounters, namely by considering more encounters from the core to the edge of the cluster, the chance of detectability increases.
- A more robust computation is needed to model these backgrounds.
- Detection of these encounters, in conjunction with observations of binary mergers, can help us constrain characteristics of BH populations.