The Role of Projective Symmetries in Quantum Metric-Affine Theory of Gravity

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Introduction

1. MAG is a natural generalization of the ideas of A. Einstein and D. Hilbert;
2. MAG is the most suitable framework for describing gravitational interaction within the gauge approach to interaction carrier fields;
3. MAG is a logical step toward constructing a potentially renormalizable theory of gravity, analogous to the transition from Fermi's four-fermion theory to the electroweak interaction theory.

Presentation Focus

The Study of Quantum Aspects of Introducing Additional Projective Symmetries and Gauge Fixing in Metric-Affine Gravity Theory.

Baikov P., Hayashi M., Nelipa N., Ostapchenko S. Ghost- and tachyon-free gauge-invariant, Poincaré, affine and projective Lagrangians // Gen. Relat. Gravit. 1992. vol. 24, p. 867–880

Symmetries in MAG (1)

$$x^m \to x'^m(x)$$

$$g_{mn} \to g'_{mn} = g_{ab} \frac{\partial x^a}{\partial x'^m} \frac{\partial x^b}{\partial x'^n}$$

$$\Gamma^l{}_{mn} \to \Gamma'^l{}_{mn} = \frac{\partial x'^l}{\partial x^r} \frac{\partial x^s}{\partial x'^m} \frac{\partial x^t}{\partial x'^n} \Gamma^r{}_{st} + \frac{\partial x'^l}{\partial x^r} \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}$$

Symmetries in MAG (2)

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = f(\lambda)\frac{dx^i}{d\lambda}$$

- If $f(\lambda) = 0$, then λ is an *affine parameter*.
- If $f(\lambda) \neq 0$, one can always perform a reparametrization $\lambda \mapsto \tilde{\lambda}(\lambda)$ to absorb $f(\lambda)$ and bring the equation back to affine form.

$$\Gamma^l{}_{mn} \to \Gamma^l{}_{mn} + N^l{}_{mn}$$

$$N^l{}_{mn} = \delta^l_m C_n + \delta^l_n B_m + F^l g_{mn} + g^{ls}E_{smn} + g^{ls}U_{smn}$$

Symmetries in MAG (3)

$$\Gamma^l{}_{mn} \to \Gamma^l{}_{mn} + N^l{}_{mn}$$

$$N^l{}_{mn} = \delta^l_m C_n + \delta^l_n B_m + F^l g_{mn} + g^{ls} E_{smn} + g^{ls} U_{smn}$$

- $E_{smn} = E_{[smn]}$
- $U_{smn} = U_{s[mn]}$
- $U_{sm}{}^m = 0$, $U_{nm}{}^n = 0$, $U^m{}_{mn} = 0$
- $U_{lmn} + U_{nlm} + U_{mnl} - U_{lnm} - U_{mln} - U_{nml} = 0$

Action of the Considered Theory

$$S = \frac{1}{\varkappa^2}\int \mathrm{d}^4 x\sqrt{-g}\{R - 2\Lambda + Q^a{}_{bc}Q^l{}_{mn}\mathcal{I}_a{}^{bc}{}_l{}^{mn} +$$
$$+ W_{abc}\,Q^l{}_{mn}\mathcal{J}^{abc}{}_l{}^{mn} +$$
$$+ W_{abc}\,W_{lmn}\,\mathcal{K}^{abclmn}\}$$

$W_{abc} = \nabla_a g_{bc} \neq 0$ - non-metricity, $\quad Q^a{}_{bc} = \Gamma^a{}_{[bc]}$ - torsion

$\left\{{}^{\,l}_{mn}\right\} = \frac{1}{2}g^{kl}\left(\partial_m g_{kn} + \partial_n g_{km} - \partial_k g_{mn}\right)$ - Levi-Civita connection

$D^l{}_{mn} = \Gamma^l{}_{mn} - \left\{{}^{\,l}_{mn}\right\} \neq 0$ - connection defect

$$\mathcal{I}_a{}^{bc}{}_l{}^{mn} = a_1 g_{al} g^{cn} g^{bm} + a_2 \delta^c_a g^{bm}\delta^n_l + a_3\delta^n_a\delta^c_l g^{bm}$$
$$\mathcal{J}^{abc}{}_l{}^{mn} = c_1 g^{am} g^{bc}\delta^n_l + c_2 g^{ac} g^{bm}\delta^n_l + c_3 g^{am}\delta^b_l g^{cn}$$
$$\mathcal{K}^{abclmn} = b_1 g^{al} g^{cn} g^{bm} + b_2 g^{am} g^{bl} g^{cn} + b_3 g^{al} g^{mn} g^{bc} +$$
$$+ b_4 g^{ac} g^{bm} g^{ln} + b_5 g^{am} g^{bc} g^{ln}$$

Generalized Projective Invariance (1)

$$\Gamma^l{}_{mn} \to \Gamma^l{}_{mn} + N^l{}_{mn}$$
$$N^l{}_{mn} = \delta^l_m C_n + \delta^l_n B_m + F^l g_{mn} + g^{ls} E_{smn} + g^{ls} U_{smn}$$

$$S[\Gamma, g] \to S[\Gamma, g] + \Delta S[\Gamma, g, C, B, F, E, U]$$
$$\Delta S = 0 \quad \Rightarrow$$

Using b_3 and c_1 as free parameters:

$$a_1 = 2 + \tfrac{3}{2}c_1,\ a_2 = 1 - \tfrac{3}{2}c_1,\ a_3 = 1 + \tfrac{3}{2}c_1,$$
$$b_1 = \tfrac{1}{4} - 6b_3 - \tfrac{1}{2}c_1,\ b_2 = -\tfrac{3}{2} - 12b_3 - \tfrac{5}{2}c_1,\ b_4 = 4b_3 + \tfrac{1}{2}c_1,$$
$$b_5 = \tfrac{1}{2} + 4b_3 + c_1,\ c_2 = -c_1,\ c_3 = -4 - 3c_1$$

Generalized Projective Invariance (2)

	a_1	b_1	b_2
a_2	$-3/2$	0	0
a_3	$-1/2$	0	0
b_3	0	-4	0
b_4	0	0	-1
b_5	0	$-1/2$	-2
c_1	-4	$-3/8$	0
c_2	-1	0	$-3/8$
c_3	-1	$-1/8$	$1/8$

Table: N = C

	a_1	a_2	b_1	b_2	b_5	c_2	c_3
1	1	2	$1/4$	$-3/2$	$1/2$	0	-4
a_3	1	-1	0	0	0	0	0
b_3	0	0	-2	8	-4	0	0
b_4	0	0	-1	-5	2	0	0
c_1	0	0	0	0	0	-1	-3

Table: N = C + B + F + E

Spin-Parity Decomposition of Fields (1)

Percacci R., Sezgin E. New class of ghost- and tachyon-free metric-affine gravities // Phys. Rev. D. 2020. vol. 101.

There are 116 projectors of this kind!

$$
\begin{aligned}
[P(2-)_{11}]^{abcdef} = & \frac{1}{3}\frac{1}{-2+d}(T^{ad}T^{ef}T^{bc} + T^{ac}T^{bd}T^{ef} + T^{ab}T^{df}T^{ce} + T^{ab}T^{cf}T^{de})- \\
& -\frac{1}{6}\frac{1}{-2+d}(T^{ac}T^{be}T^{df} + T^{af}T^{bc}T^{de} + T^{ae}T^{bc}T^{df} + T^{ac}T^{de}T^{bf})- \\
& -\frac{2}{3}\frac{1}{-2+d}T^{ab}T^{cd}T^{ef}+ \\
& +\frac{1}{3}T^{af}T^{cd}T^{be} + \frac{1}{3}T^{ae}T^{cd}T^{bf}- \\
& -\frac{1}{6}T^{ad}T^{ce}T^{bf} - \frac{1}{6}T^{af}T^{bd}T^{ce} - \frac{1}{6}T^{ae}T^{bd}T^{cf} - \frac{1}{6}T^{ad}T^{cf}T^{be}
\end{aligned}
$$

$$T_a^b = \delta_a^b - \hat{k}_a\hat{k}^b$$

Spin-Parity Decomposition of Fields (2)

Found new ghost- and tachyon-free theories with restrictions:

1. The spin-3 field does not propagate.
2. In the spin-2^+ sector, only the massless graviton propagates.

Percacci R., Sezgin E. New class of ghost- and tachyon-free metric-affine gravities // Phys. Rev. D. 2020. vol. 101.

Gauge Fixing and Existence of the Propagator

Gauge fixing: $\gamma_{abc} \to \gamma_{[abc]}$

$$
\begin{aligned}
(\mathcal{X}^{-1})_{[lmn][xyz]} = \frac{1}{12(1-a_1+a_3)} \times \\
\times \big(g_{lx} g_{nz} g_{my} + g_{lz} g_{xm} g_{ny} + g_{ly} g_{nx} g_{mz} - \\
- g_{lz} g_{xn} g_{my} - g_{ly} g_{xm} g_{nz} - g_{lx} g_{ny} g_{mz} \big)
\end{aligned}
$$

$$
\left.\begin{aligned}
N &= E \\
N &= C + B + F + U \\
N &= C + B + F + U + E
\end{aligned}\right] \Rightarrow 1 - a_1 + a_3 = 0
$$

Studied gauge choices: $\gamma_{abc} \to \gamma_{a[bc]}$, $\gamma_{abc} \to \gamma_{[ab]c}$,
$\gamma_{abc} \to \gamma_{a(bc)}$

Main Results

1. There are several fundamental ideas underlying MAG.
2. Since experiments on quantum gravity are not feasible, we must rely on theoretical considerations to reduce the number of independent parameters of the theory.
3. These approaches require extensive symbolic computations, which have only recently become accessible to small research groups.
4. For this reason, a comprehensive study of the renormalizability of quantum MAG may be within reach in the near future.

Thank you for your attention!

Appendix: Design of the Symbolic Computation Toolkit