Petr I. Pronin
Moscow State University

Faculty of Physics

QUANTIZATION OF AFFINE-METRIC GRAVITATIONAL THEORY

Succession of action

The main problems to construct the quantum description of fundamental interactions

STEPS

- A choice of dynamical variables
- Symmetries of fields
- Analysis of measurability of fields.
- Divergencies in S-matrix.
- Renormalizability of model
- Theorem of equivalence and correspondence

Good Gauge Theories

Satisfactory solutions of the aboveordered questions have been proposed in QED and QCD

Material fields $\{\varphi_i(x)\}$ - values of space of fundamental representation of G-group

Symmetries $\varphi_i(x) \quad \rightarrow \quad {}'\varphi_i(x) = T_i^j \varphi_j(x)$

Material fields are transformed on fundamental representation group G

Gauge fields (elctromagnetic and Yang-Miills fields) - (compensating fields).

Symmetries $A(x) \quad \rightarrow \quad {}'A(x) = \theta A(x) \theta^{-1} + \theta d \theta^{-1}$

Gauge fields are transformed on adjoin representation of group G

The canonical dimensions of material fields in atomic system $\hbar = c = 1$ and $dim(\mathfrak{M}) = 4$

$[\varphi(x)] = [m^{-1}] \equiv [-1], \quad [\psi(x)] = [m^{-\frac{3}{2}}] \equiv [-\frac{3}{2}]$

The canonical dimensions of gauge fields

$[A(x)] = [m^{-1}] \equiv [-1]$

GAUGE THEORIES IS THE PHYSICAL THEORIES IN WHICH THE GAUGE FIELD IS A CONNECTION ON PRINCIPAL BUNDLE

Base manifold - Riemannian manifold

Symmetries - general coordinate trasformations
$x_\mu \Rightarrow x'_\mu = x'_\mu(x)$

Material fields are transformed as

$$\Phi(x)^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_k} \Rightarrow \Phi'(x')^{\alpha'_1\ldots\alpha'_n}_{\beta'_1\ldots\beta'_k} = \frac{\partial x'^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\alpha_n}}{\partial x^{\alpha_n}} \cdot \frac{\partial x^{\beta_1}}{\partial x^{\beta'_1}} \cdots \frac{\partial x^{\beta_k}}{\partial x^{\beta'_k}} \Phi(x)^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_k}$$

$$g'_{\mu\nu}(x') = \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}(x)$$

Canonical dimension of metric field is equal 0

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \Rightarrow [g_{\mu\nu}] = [m^0] \equiv [0]$$

GR is the gauge theory of the general coordinate transformations group -

IS IMPROPER

Divergences and Renormalizability

- QED and QCD are renomalizable in all orders.

- GR is finite at 1-loop approchimation.

- Quadratic gravitational theories The theory is renormalisable. There is the problem of unitarity. [Stelle K.S.- PRD16, (1977) h. 199-213].

2-loops counterterms are dependent on a gauge fixing and parametrization on the equation of motion.
[Kazakov K.A.and Pronin P.I- PRD59, (1999),064012]

- Hŏrava-Lifschez theory - [Hŏrava P. - PRD79, (2009), 084008]

- Theories with higher derivatives [Tombolius E. T. (1997)]

- Supergravity (supersymmetric generalization of GR)

The geometrical consequence of STR: "The Minkowskian space $\mathfrak{M}(4)$ is the affine space

The transformation group of $\mathfrak{M}(4)$ - the group of affine transfomations $GA(4, R)$ J. Beem and P. Ehrlich,. "Global Lorentzian Geometry"(1981),Marcel Dekker inc (new York)

There were 3 ways to construct of gauge theory of affine group:

- "Physical"approach
  Hehl F.W. and all, Phys. Lett B63, (1976), 446-448
- "Semi-geometrical"approach
  Lord E.A., Phys. Lett A65 (1978), 1-4
- "Mathematical"approach
  Ne'eman Y., *Šijački* ,Phys. Lett B157, (1985), 275-279

⋆ More precise definitions and descriptions $\sim$ 50-60 papers

"Mathematical"approach to gauge theory of affine group

The affine space is the set of two objects: namely - point and vectors
$A(4,R)$ - affine space
$\xi \in R(4)$ where $R(4)$ is the vector group
Then the isomorphism $R(4)$ into $A(4,R)$

$$\xi \to \begin{bmatrix} I_4 & \xi \\ 0 & 1 \end{bmatrix} \in A(4,R) \ni \begin{bmatrix} I_4 & \xi \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} \mathfrak{A} & \xi \\ 0 & 1 \end{bmatrix} \in GA(4,R)$$

The connection (gauge field) in the fibre bundle $\mathcal{E}(M, R(5), \pi, G)$

$$\Gamma = \begin{bmatrix} \omega & h \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad {}'\Gamma = \mathfrak{A}\Gamma\mathfrak{A}^{-1} + \mathfrak{A}d\mathfrak{A}^{-1}$$

$$\mathfrak{A} = \begin{bmatrix} \mathrm{A} & \mathrm{h} \\ 0 & 1 \end{bmatrix} \quad \Leftrightarrow \quad \mathfrak{A}^{-1} = \begin{bmatrix} A^{-1} & A^{-1}e \\ 0 & 1 \end{bmatrix}$$

but **h** is not the tetrads and only "soldering"form

Non-linear realization of the affine group

Based on

Isham C.J., Salam A., Strathdee J., Ann. Phys. **62**, (1971), 98

Borisov A.B., Ogievetsky V.I., Theor. Math. Phys.,**21**(1974), 329; 1179

Lord E.A., J. Math. Phys.**27**(1986), 2415

Cho Y.M., Phys. Rev.D,**18**,(1978), 2810

Lie algebra $\mathfrak{G} = \mathcal{H} \oplus V$

$\mathcal{H}$ is the Lie algebra of subgroup $H \subset G$

Then it is possible

$$\Gamma = \hat{\Gamma} + B$$

$$\hat{\Gamma} \quad \to \quad \hat{\Gamma}' = h\hat{\Gamma} h^{-1} + hdh^{-1}$$

$$B \quad \to \quad B' = hBh^{-1}$$

The choice $h \in \mathcal{H}$ - infinitesimal general coordinate transformations

$$\hat{\Gamma} \sim \Gamma^c{}_{ab} \Rightarrow \Gamma'^c{}_{ab} = \Gamma^k{}_{mn} \frac{\partial x'^c}{\partial x^k} \frac{\partial x^m}{\partial x'^a} \frac{\partial x^n}{\partial x'^b} + \frac{\partial x'^c}{\partial x^i} \frac{\partial^2 x^i}{\partial x'^a \partial x'^b}$$

$$B \sim g_{mn} \Rightarrow g'_{mn} = g_{ij} \frac{\partial x^i}{\partial x'^m} \frac{\partial x^j}{\partial x'^n}$$

THIS IS THE BASE OF AFFINE-METRIC GAUGE THEORY

Based on
Krupka D., *Janyška* J., (1990) Lectures on differential invariants
G - Lie group on manifold $\mathcal{M}$
Mapping $G \times \mathcal{M} \ni (g, x) \to g * x \in \mathcal{M}$ is the action G on $\mathcal{M}$.
Functions $f : \mathcal{M} \to \mathcal{L}$ are G invariant if and only if если $f(g * x) = f(x)$ for all $x \in \mathcal{M}$ and all и $g \in G$.
Let us consider the geometrical objects on affine manifold as the coordinates $\mathcal{T}^1_n\mathcal{M}$

- $g_{ab} = g_{ba}; \qquad W_{cab} \equiv \nabla_c g_{ab}$
- $Q^c{}_{ab} = \frac{1}{2}(\Gamma^c{}_{ab} - \Gamma^c{}_{ba}); \qquad S^c{}_{ab} = \frac{1}{2}(\Gamma^c{}_{ab} + \Gamma^c{}_{ba})$
- $R^c{}_{akb} = \Gamma^c{}_{ab,k} - \Gamma^c{}_{ak,b} + \Gamma^c{}_{nk}\Gamma^n{}_{ab} - \Gamma^c{}_{an}\Gamma^n{}_{kb};$
- $V^c{}_{abn} = Q^c{}_{ab,n};$
- $U^a{}_{mnb} = \frac{1}{3}\left(S^a{}_{mn,b} + S^a{}_{nb,m} + S^a{}_{bm,n}\right).$
- $\Omega_{mn} = \frac{\partial \Gamma^c{}_{cn}}{\partial x^m} - \frac{\partial \Gamma^c{}_{cm}}{\partial x^n}.$

Algorithm to construct of the AMTG Lagrangian

LAMGT = SYM of INVARIANTS

$$\left\{\Sigma + \Sigma_{\alpha}{}^{\beta} + \Sigma^{\beta}{}_{\alpha}{}^{\gamma} + \Sigma_{\alpha}{}^{\beta\gamma\lambda}\right\}\mathcal{L}(g, \partial g, \Gamma, \partial\Gamma) = 0 \qquad (\star)$$

where

$$\Sigma = \Omega_{\mu\nu}\frac{\partial}{\partial\Omega^{\mu\nu}}, \quad \Sigma^{\beta}{}_{\alpha}{}^{\gamma} = \frac{\partial}{\partial S^{\alpha}{}_{\beta\gamma}}, \quad \Sigma_{\alpha}{}^{\beta\gamma\lambda} = \frac{\partial}{\partial U^{\alpha}{}_{\beta\gamma\lambda}}$$

$$\begin{aligned}\Sigma_{\alpha}{}^{\beta} = {} & 2g^{\nu\beta}\frac{\partial}{\partial g^{\nu\alpha}} + 2W^{\beta}{}_{\nu\mu}\frac{\partial}{\partial W^{\alpha}{}_{\mu\nu}} - W^{\mu\nu}{}_{\alpha}\frac{\partial}{\partial W^{\mu\nu}{}_{\beta}} + \\ & + Q^{\beta}{}_{\mu\nu}\frac{\partial}{\partial Q^{\alpha}{}_{\mu\nu}} - 2Q^{\nu}{}_{\alpha\mu}\frac{\partial}{\partial Q^{\nu}{}_{\beta\mu}} + \\ & + R^{\beta}{}_{\mu\nu\lambda}\frac{\partial}{\partial R^{\alpha}{}_{\mu\nu\lambda}} - R^{\mu}{}_{\alpha\nu\lambda}\frac{\partial}{\partial R^{\mu}{}_{\beta\nu\lambda}} - 2R^{\mu}{}_{\nu\alpha\lambda}\frac{\partial}{\partial R^{\mu}{}_{\nu\beta\lambda}} + \\ & + V^{\beta}{}_{\mu\nu\lambda}\frac{\partial}{\partial V^{\alpha}{}_{\mu\nu\lambda}} - 2V^{\mu}{}_{\alpha\nu\lambda}\frac{\partial}{\partial V^{\mu}{}_{\beta\nu\lambda}} - V^{\lambda}{}_{\mu\nu\alpha}\frac{\partial}{\partial V^{\lambda}{}_{\mu\nu\beta}}\end{aligned}$$

General solution of equation $(\star)$ contain 246 terms !!??

Lagrangian of AMTG

$$\mathcal{S}_{tot} = \int d^4x\sqrt{-g}\Big\{ -\frac{1}{\varkappa^2}\Big[2\Lambda - R(g,\Gamma) - Q^c{}_{ab}Q^k{}_{mn}\mathcal{I}^{abmn}_{ck} +$$

$$+W_{cab}W_{kmn}\mathcal{K}^{abckmn} + W_{kmn}Q^c{}_{ab}\mathcal{J}^{abmn}_c\Big] + R^c{}_{asb}R^k mtn\mathcal{N}^{asbmtn}_{ck}\Big\}$$

$$\mathcal{I}^{abmn}_{ck} = a_1 g_{cl}g^{am}g^{bk} + a_2 g^{am}\delta^b_c\delta^k_l + a_3 g^{am}\delta^b_l\delta^c_k,$$

$$\mathcal{K}^{kmncab} = b_1 g^{ck}g^{am}g^{bn} + b_2 g^{cm}g^{ak}g^{bn} + b_3 g^{mn}g^{ab}g^{ck} + b_4 g^{am}g^{cb}g^{kn} +$$

$$+b_5 g^{ab}g^{cm}g^{kn}$$

$$\mathcal{J}^{kmnab}_c = c_1\delta^b_c g^{mn}g^{ka} + c_2\delta^b_c g^{ma}g^{kn} + c_3\delta^m_c g^{ak}g^{bn}$$

$\mathcal{N}^{asbmtn}_{ck}$ is the sum 17 term these are the combination like this $g_{ck}g^{am}g^{st}g^{bn}$

28 UNKNOWN CONSTANTS

$$a_i, \quad i = 1,2,3; \qquad b_j, \quad j = 1,...,5; \qquad c_k, \quad k = 1,2,3;$$

$$d_n, \quad n = 1,.......,17$$

Motivation of restriction

* Phenomenological reason

* Unitarity

* New algebraic symmetries

* Renormalizability

Ghost- and tachions free Lagrangians in AMTG

Baikov P., Hayashi M., Nelipa N., Ostapchenko (1992), GRG, v. 24, p. 867
Percacci R., Sezgin E. (2020), PRD101,084040
Marzo C. (2022) PRD105, 065017
Linearization of gravitational action by means

$$g_{ab} = \eta_{ab} + \varkappa h_{ab}(x), \qquad \Gamma^c{}_{ab} = \varkappa \gamma^c{}_{ab}$$

allow us to find the relations for unknown constants due to the propagator for h and γ

$$D_{hh}(k), \qquad D_{\gamma\gamma}(k)$$

are free from ghost- and tachions
1.The method give us the restrictions on unknown constants but does not give us rules for exclusion of any terms
2. Does not guarantee of renormalzability

$$\triangle\mathcal{S}_{1-loop} = -i\int d\mu(h)d\mu(\gamma)\exp\Big(-i\int d^4x\sqrt{-g}\mathcal{L}_2\Big)$$

Structure of $\mathcal{L}_2$ is

$$\mathcal{L}_2(h,\gamma,g,\Gamma) = \nabla_k h^{xy}\mathcal{D}^{kr}_{xyzv}\nabla_r h^{zv} + h^{xy}\mathcal{M}_{xyzv}h^{zv} + \gamma^c{}_{ab}\mathcal{X}^{abmn}_{ck}\gamma^k{}_{mn} +$$

$$+\gamma^c{}_{ab}\mathcal{A}^{ab}_{cxy}h^{xy} + \gamma^c{}_{ab}\mathcal{B}^{abt}_{cxy}\nabla_t h^{xy} + \nabla_k h^{xy}\mathcal{O}^k_{xyzv}h^{zv}$$

$$\mathcal{D}^{kr}_{xyzv} = \Pi_{xymn}\mathcal{K}^{kmnrpq}\Pi_{zvpq}, \qquad \Pi_{ijst} = \frac{1}{2}(g_{is}g_{jt} + g_{js}g_{it})$$

Omitting the indexes

$$\mathcal{L}^{(2}(h,\gamma,g,\Gamma) = \nabla h\mathcal{D}\nabla h + h\mathcal{M}h + h\mathcal{O}\nabla h + \gamma\mathcal{X}\gamma + \gamma\mathcal{A}h + \gamma\mathcal{B}\nabla h.$$

AMTG of Hilbert-Einstein type

Kalmykov M.Yu. and Pronin P.I. (1991), One loop effective action in gauge gravitational theory, Nuovo Cmento, B106, p.1401-1415;

$$\mathcal{L} = \sqrt{-g}(R(g,\Gamma) - 2\Lambda)$$

Lagrangian quadratic on metric and connection

$$\mathcal{L}^{(2}(h,\gamma,g,\Gamma) = h\mathcal{M}h + \gamma\mathcal{X}\gamma + \gamma\mathcal{A}h + \gamma\mathcal{B}\nabla h.$$

is invraiant with respect general coordinate and projective transformatinos

$$\gamma^c{}_{ab} \rightarrow {}'\gamma^c{}_{ab} = \gamma^c{}_{ab} + \delta^c_a C_b$$

Propagator of quantum field is not defined
It is need to fix projective invariance

$$\mathcal{L}_{gf} = \frac{1}{2} f_n f^n, \qquad f_n = \left(p g_{cn} g^{ab} + q\delta^a_c\delta^b_n + s\delta^a_n\delta^b_c\right)\gamma^c{}_{ab}$$

We use the dimension regularization

$$\triangle\mathcal{S}_{1-loop} = \frac{1}{32\pi^2(n-4)}\int d^4x\sqrt{-g}\frac{58}{5}\Lambda^2$$

There is the asumptotic freedom for Λ

$$\beta(\Lambda) = -\frac{29}{160\pi^2}\Lambda^2$$

Alternative calculation of 1-loop divergences in AMTΓ of H-E

Without index form

$$\mathcal{L}^{(2}(h, \gamma, g, \Gamma) = h\mathcal{M}h + \gamma\mathcal{X}\gamma + \gamma\mathcal{A}h + \gamma\mathcal{B}\nabla h$$

Excluding total divergence

$$\mathcal{L}^{(2}(h, \gamma, g, \Gamma) = h\mathcal{M}h + \gamma\mathcal{X}\gamma + \gamma\hat{\mathcal{A}}h + \nabla\gamma\hat{\mathcal{B}}h$$

we may use the new variables

$$h \quad \rightarrow \quad h = h' + \omega(\gamma, \nabla\gamma, W, Q, g, \Gamma) \Rightarrow d\mu(h) = d\mu(')$$

and then the answer will be

$$\mathcal{L}_2 \rightarrow \check{\mathcal{L}}_2 = \nabla\gamma\check{\mathcal{D}}_{(\gamma\gamma)}\nabla\gamma + \gamma\mathcal{X}\gamma + h'\check{\mathcal{M}}h'$$

Exeptions of calculation

A.Yu.Baurov, P.I.Pronin, K.V.Stepanyantz Class. Quantum Grav. 35 (2018) 085006

$$\breve{\mathcal{D}}_{ck}^{abmn} = T_c{}^{abpq} * T_k{}^{mn}{}_{pq}$$

where

$$T^c{}_{abpq} = \frac{1}{2}\delta_a^c \left(g_{bq}\nabla_p + g_{bp}\nabla_q\right) + \frac{1}{2}\left(g_{ab}g_{pq} - g_{ap}g_{bq} - g_{aq}g_{bp}\right)$$

Tt is need to choose the gauge in that we extract the components of γ that are out of action of $T^c{}_{abpq}$ and put these components are equal zero
The result

$$\triangle \mathcal{S}_{1-loop} = \frac{1}{32\pi^2(n-4)} \int d^4x\sqrt{-g}\frac{58}{5}\Lambda^2$$

$$\mathcal{S} = \frac{1}{\varkappa^2} \int d^4x \sqrt{-g} \Big\{ 2\Lambda + R(g, \Gamma) \Big\}$$
$$+ Q^c{}_{ab} Q^k{}_{mn} \mathcal{I}^{abmn}_{ck} \Big\}$$

There are many projective symmetries that are to be fixed

$$\Gamma^c{}_{ab} \to {}'\Gamma^c{}_{ab} = \Gamma^c{}_{ab} + p\delta^c_a C_b + qg^{cs} I_{[sab]} + tV^c{}_{ab}$$

General case

$$\mathcal{S}_2 = \int d^4x\sqrt{-g}\left\{\frac{1}{\varkappa^2}\Big[2\Lambda + R(g,\Gamma)\Big] + Q\mathcal{I}Q + W\mathcal{K}W + Q\mathcal{J}W + R\mathcal{T}R\right\}$$

$$\mathcal{L}_2(h,\gamma,g,\Gamma) = \nabla_k h^{xy}\mathcal{D}^{kr}_{xyzv}\nabla_r h^{zv} + h^{xy}\mathcal{M}_{xyzv}h^{zv} + \nabla_k h^{xy}\mathcal{O}^{k}_{xyzv}h^{zv} +$$

$$+\nabla_s\gamma^c{}_{ab}\mathcal{T}^{sabtmn}_{ck}\nabla_t\gamma^k{}_{mn} + \gamma^c{}_{ab}\mathcal{X}^{abmn}_{ck}\gamma^k{}_{mn} + \nabla_s\gamma^c{}_{ab}\mathcal{S}^{sabmn}_{ck}\gamma^k{}_{mn}$$

$$+\gamma^c{}_{ab}\mathcal{A}^{ab}_{cxy}h^{xy} + \gamma^c{}_{ab}\mathcal{B}^{abt}_{cxy}\nabla_t h^{xy}$$

The goal

$$\gamma^c{}_{ab} \to \gamma'^c{}_{ab} = \gamma^c{}_{ab} + \omega^c{}_{ab}$$

$$\mathcal{L}_2(h,\gamma,g,\Gamma) \to \mathcal{L}_2(h,\gamma',g,\Gamma) =$$

$$= \nabla_k h^{xy}\mathcal{D}^{kr}_{xyzv}\nabla_r h^{zv} + h^{xy}\mathcal{M}_{xyzv}h^{zv} + \nabla_k h^{xy}\mathcal{O}^{k}_{xyzv}h^{zv} +$$

$$+\nabla_s\gamma^c{}_{ab}\mathcal{T}^{sabtmn}_{ck}\nabla_t\gamma^k{}_{mn} + \gamma^c{}_{ab}\mathcal{X}^{abmn}_{ck}\gamma^k{}_{mn} + \nabla_s\gamma^c{}_{ab}\mathcal{S}^{sabmn}_{ck}\gamma^k{}_{mn}$$

Result

$$\omega^c{}_{ab}(x) = \int d^4y \mathfrak{L}^c{}_{ab}{}^{kmn}(x-y)\nabla_k h_{mn}(y) + \int d^4y \mathfrak{P}^c{}_{ab}{}^{kmn}(x-y) h_{mn}(y)$$

The exact result has been get for

$$\mathcal{S}_{tot} = \int d^4x\sqrt{-g}\Big\{-\frac{1}{\varkappa^2}\Big[2\Lambda - R(g,\Gamma) + -Q^c{}_{ab}Q^k{}_{mn}\mathcal{I}^{abmn}_{ck}\Big] + \xi R^2(g,\Gamma)\Big\}$$

Poslavsky S.V.,Pronin P.I. (2015)

THANK YOU ON THE ATTENTION!