Photon sphere and quasinormal modes for a dyonic dilatonic black hole

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based on the papers:

A. N. Malybayev, K. A. Boshkayev, and V. D. I., Eur. Phys. J. C **81**, 475 (2021).
K. Boshkayev, G. Takey, V. I., A. Malybayev, G. Nurbakova, A. Urazalina, Phys. of the Dark Universe, v. 48, 101862 (2025); arXiv: 2411.15006.

Motivations

- the detection of gravitational waves
- discovery of compact objects, e.g. like supermassive black hole in at the heart of the M87 galaxy and our Milky Way galaxy
- exploring of black hole shadows, photon spheres, accretion discs, ISCO (innermost stable circular orbits) etc
- modified gravitational theories
- the study of strong field regime in gravity
- supergravities, superstrings
- black hole/brane solutions with several charges related to Lie algebras, polynomials and Toda chains

[V.D.I., Black brane solutions governed by fluxbrane polynomials, J. Geom. Phys., 86, 101-111 (2014); arXiv:1401.0215.]

The action of the model

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\Big\{R[g] - g^{\mu\nu}\partial_\mu\vec{\varphi}\partial_\nu\vec{\varphi} - \frac{1}{2}e^{2\vec{\lambda}_1\vec{\varphi}}F^{(1)}_{\mu\nu}F^{(1)\mu\nu} - \frac{1}{2}e^{2\vec{\lambda}_2\vec{\varphi}}F^{(2)}_{\mu\nu}F^{(2)\mu\nu}\Big\}, \qquad (2.1)$$

where $g = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$ is the metric, $|g| = |\det(g_{\mu\nu})|$, $\vec{\varphi} = (\varphi^1, \varphi^2)$ is the vector of two scalar fields, $F^{(i)} = dA^{(i)} = \frac{1}{2}F^{(i)}_{\mu\nu}dx^\mu \wedge dx^\nu$ is the 2-form with $A^{(i)} = A^{(i)}_\mu dx^\mu$, $i = 1,2$; G is the gravitational constant, $\vec{\lambda}_1 = (\lambda_{1i}) \neq \vec{0}$, $\vec{\lambda}_2 = (\lambda_{2i}) \neq \vec{0}$ are the dilatonic coupling vectors, which obey

$$\vec{\lambda}_1 \neq -\vec{\lambda}_2, \qquad (2.2)$$

and $R[g]$ is the Ricci scalar. Here, we set $c = 1$.

Dyonic-like black hole solution

The BH solution [A.N. Malybayev, K.A. Boshkayev, V.D.I., Eur. Phys. J. C **81**, 475 (2021)] to the action (2.1) is defined on the manifold $\mathcal{M} = \mathbb{R} \times (2\mu, +\infty) \times S^2$, and has the following form

$$\begin{aligned} ds^2 &= H^a \Big\{ -H^{-2a} \left(1 - \frac{2\mu}{R}\right) dt^2 \\ &\qquad + \frac{dR^2}{1 - \frac{2\mu}{R}} + R^2 d\Omega^2 \Big\}, \qquad (2.3) \\ \varphi^i &= \nu^i \ln H, \qquad (2.4) \end{aligned}$$

with the 2-form defined by

$$F^{(1)} = \frac{Q_1}{H^2 R^2} dt \wedge dR, \quad F^{(2)} = Q_2 \tau, \qquad (2.5)$$

where Q_1 is (color) electric charge and Q_2 is (color) magnetic one.

Here $\mu > 0$, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, where $0 < \theta < \pi$ and $0 < \phi < 2\pi$, $\tau = \sin\theta d\theta \wedge d\phi$ is volume form on S^2.
The moduli function is defined as

$$H = 1 + \frac{P}{R}. \tag{2.6}$$

The parameter $P > 0$ obeys

$$P(P + 2\mu) = \frac{1}{2}Q^2, \tag{2.7}$$

or, equivalently

$$P = -\mu + \sqrt{\mu^2 + \frac{1}{2}Q^2}. \tag{2.8}$$

The parameters of the solution are defined as follows

$$a = \frac{(\vec{\lambda}_1 + \vec{\lambda}_2)^2}{\Delta}, \tag{2.9}$$

$$\Delta \equiv \frac{1}{2}(\vec{\lambda}_1 + \vec{\lambda}_2)^2 + \vec{\lambda}_1^2 \vec{\lambda}_2^2 - (\vec{\lambda}_1 \vec{\lambda}_2)^2, \tag{2.10}$$

$$\nu^i = \frac{\lambda_{1i} \vec{\lambda}_2 (\vec{\lambda}_1 + \vec{\lambda}_2) - \lambda_{2i} \vec{\lambda}_1 (\vec{\lambda}_1 + \vec{\lambda}_2)}{\Delta}, \tag{2.11}$$

$i = 1, 2$ and

$$Q_1^2 = \frac{\vec{\lambda}_2 (\vec{\lambda}_1 + \vec{\lambda}_2)}{2\Delta} Q^2, \quad Q_2^2 = \frac{\vec{\lambda}_1 (\vec{\lambda}_1 + \vec{\lambda}_2)}{2\Delta} Q^2. \tag{2.12}$$

Here, the additional constraints are imposed

$$\vec{\lambda}_1 (\vec{\lambda}_1 + \vec{\lambda}_2) > 0, \qquad \vec{\lambda}_2 (\vec{\lambda}_1 + \vec{\lambda}_2) > 0. \tag{2.13}$$

Note that the restrictions (2.13) imply inequalities $\vec{\lambda}_s \neq \vec{0}$, $s = 1, 2$, and (2.2).

It can be readily verified that

$$\Delta > 0, \tag{2.14}$$

is valid for $\vec{\lambda}_1 \neq -\vec{\lambda}_2$, since $\frac{1}{2}(\vec{\lambda}_1 + \vec{\lambda}_2)^2 > 0$ and

$$C = \vec{\lambda}_1^2 \vec{\lambda}_2^2 - (\vec{\lambda}_1 \vec{\lambda}_2)^2 \geq 0, \tag{2.15}$$

because of the Cauchy–Schwarz inequality. Here, $C = 0$ if and only if vectors $\vec{\lambda}_1$ and $\vec{\lambda}_2$ are collinear. Relation (2.15) implies

$$0 < a \leq 2. \tag{2.16}$$

For non-collinear vectors $\vec{\lambda}_1$ and $\vec{\lambda}_2$ we find $0 < a < 2$, and for collinear ones we get $a = 2$.

The gravitational mass (for $G = 1$) reads:

$$M = \mu + \frac{a}{2} P. \tag{2.17}$$

The mass M and charge Q obey the inequality [MBI]

$$\frac{Q^2}{M^2} < \frac{8}{a^2}. \tag{2.18}$$

Defining eqs. (2.9)-(2.12) imply

$$\vec{\nu}^2 = \frac{a(2-a)}{2}, \tag{2.19}$$

$$Q_1^2 + Q_2^2 = \frac{a}{2} Q^2. \tag{2.20}$$

For vanishing $a \to 0$ we get $\nu^i \to 0$, $Q_i \to 0$ and $\varphi^i \to 0$ and ds^2 reduces to the Schwarzschild metric.

In the case $a = 2$ we get the Reissner–Nordström metric.

Geodesic equations

The eqs. for geodesics $x^\alpha(\tau)$ are derived from the Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta. \tag{2.21}$$

They are equivalent to the Euler-Lagrange equations:

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0. \tag{2.22}$$

Here $\dot{x}^\alpha = dx^\alpha / d\tau = u^\alpha$ is the 4-velocity vector, τ is affine parameter for null geodesics and the proper time for a massive point-like particle, respectively, $\alpha = 0, 1, 2, 3$.
For conserved energy we get

$$\mathcal{E} = \mathcal{L} = \frac{1}{2} g_{\alpha\beta}(x) u^\alpha u^\beta = -k/2. \tag{2.23}$$

Here, we put $k = 0, 1$, which corespond to null, and timelike geodesics, respectively.

Integrals of motion

We put $\theta = \pi/2$. For our metric the (reduced) Lagrangian is:

$$\mathcal{L}_* = \frac{1}{2} H^a \left[-H^{-2a} \left(1 - \frac{2\mu}{R} \right) \dot{t}^2 + \frac{\dot{R}^2}{1 - \frac{2\mu}{R}} + R^2 \dot{\phi}^2 \right] . \quad (2.24)$$

Given cyclic coordinates t (time) and ϕ (angle), the system possesses conserved quantities described by the integrals of motion

$$\tilde{E} = H^{-a} \left(1 - \frac{2\mu}{R} \right) \dot{t}, \quad \tilde{L} = H^a R^2 \dot{\phi}, \quad (2.25)$$

related for $k = 1$ to the total energy $E = \tilde{E}m$ and angular momentum $L = \tilde{L}m$, respectively, of a test (neutral) point-like particle of mass m.

Effective potential

For the line element from Eq. (2.3) we get

$$-H^{-a}\left(1-\frac{2\mu}{R}\right)\dot{t}^2+\frac{H^a\dot{R}^2}{1-\frac{2\mu}{R}}+H^aR^2\dot{\phi}^2=-k. \tag{2.26}$$

Using Eqs. (2.25), this relation simplifies to the following differential equation:

$$-\frac{H^a\tilde{E}^2}{\left(1-\frac{2\mu}{R}\right)}+\frac{H^a\dot{R}^2}{1-\frac{2\mu}{R}}+\frac{H^{-a}\tilde{L}^2}{R^2}=-k, \tag{2.27}$$

which can be presented in a compact form:

$$\dot{R}^2+V^2=\tilde{E}^2, \tag{2.28}$$

by using the effective potential

$$V=\sqrt{H^{-2a}\left(1-\frac{2\mu}{R}\right)\left(H^ak+\frac{\tilde{L}^2}{R^2}\right)}. \tag{2.29}$$

Radial equation

The Lagrange equation for the radial coordinate R may be presented in the following form

$$\ddot{R} + V\frac{\partial V}{\partial R} = 0. \tag{2.30}$$

For $\dot{R} \neq 0$, it follows just from Eq. (2.28).
For circular trajectory with $\dot{R} = 0$, the radial equation (2.30) reads

$$\frac{\partial V}{\partial R} = 0, \tag{2.31}$$

while equation (2.28) takes the following form

$$V^2 = \tilde{E}^2. \tag{2.32}$$

Eq. (2.31) is not a direct consequence of Eq. (2.28) and requires separate consideration.

Circular null geodesics

Here, we consider circular motions, which are described by condition: $\dot{R} = 0$ so $V = \tilde{E}$.
For $k = 0$ we get

$$
\begin{aligned}
R_0 \equiv & \frac{1}{2}\Big[P(a-1) + 3\mu \\
& + \sqrt{(P(1-a) - 3\mu)^2 - 4P\mu(2a-3)}\Big],
\end{aligned}
\tag{2.33}
$$

the radius of the photon sphere, obeying $R_0 > 2\mu$.

Black hole shadow

The standard consideration gives for the shadow angle ($k = 0$):

$$\vartheta_{sh} = \arcsin\left(\frac{V(R_{obs})}{V(R_0)}\right), \tag{2.34}$$

for all $R_{obs} > R_0$. Here, R_{obs} describes the position of an observer, and R_0 is the radius of the photon sphere.
For $R_{obs} \gg \mu$ and $R_{obs} \gg P$ we get:

$$\vartheta_{sh} = \frac{b_*}{R_{obs}} + O\left(\frac{1}{R_{obs}^2}\right), \tag{2.35}$$

as $R_{obs} \to +\infty$. Here

$$b_* = \frac{1}{V(R_0)} \tag{2.36}$$

is critical impact parameter.

Quasinormal modes

The radius of photonic sphere and the effective potential $U(R) = (V(R, \tilde{L}^2))^2$ ($k = 0$) with "quantized" $\tilde{L}^2$, i.e. replaced by $l(l+1)$,

$$U_{eik}(R) = \frac{A(R)l(l+1)}{C(R)} \tag{2.37}$$

where $l = 0, 1, 2, \ldots$, may be used for calculation of the "spectrum" of quasinormal modes (QNM) for test fields in the eikonal approximation, i.e. when $l \to +\infty$. Here $A = A(R)$ is redshift function and $C(R)$ is central function for our metric $ds^2 = -Adt^2 + A^{-1}dR^2 + Cd\Omega^2$.

The test scalar field Ψ obeys standard Klein-Fock-Gordon (KFG) equation

$$\Delta\Psi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\mu\Psi) = 0. \tag{2.38}$$

Eikonal QNM

We use the ansatz

$$\Psi = e^{-i\omega t}\Psi_*(R)\,Y_{lm}, \tag{2.39}$$

where Y_{lm} are the spherical harmonics and $\Psi_*(R)$ obey QNM boundary conditions with $|\Psi_*(R)| \to +\infty$: as $R \to 2\mu$ and $R \to +\infty$.

The asymptotic eikonal QNM "spectrum" (as $l \to +\infty$):s

$$\mathrm{Re}(\omega) = \left(l + \frac{1}{2}\right) H_0^{-a} F_0^{1/2} R_0^{-1}, \tag{2.40}$$

$$\mathrm{Im}(\omega) = -\left(n + \frac{1}{2}\right) H_0^{-a-1/2} F_0^{1/2} R_0^{-3/2} \mathcal{D}^{1/4}, \tag{2.41}$$

where

$$H_0 = 1 + \frac{P}{R_0}, \qquad F_0 = 1 - \frac{2\mu}{R_0} \tag{2.42}$$

where $\mathcal{D} = (P(1-a) - 3\mu)^2 - 4P\mu(2a-3)$. Here $n = 0, 1, 2, \ldots$ is the overtone number ($n \ll l$).

Circular geodesics for massive test particles

We now focus on timelike geodesics $(k = 1)$ with $\dot{R} = 0$. The radial equation (2.31) ($\frac{\partial V}{\partial R} = 0$) implies

$$\frac{L^2}{m^2} = \frac{H^a R^2 \left[aP(R - 2\mu) + 2\mu(P + R) \right]}{2\Delta_0}, \tag{2.43}$$

where

$$\Delta_0 = R^2 + ((1 - a)P - 3\mu)R + (2a - 3)P\mu = (R - R_0)(R - R_-), \tag{2.44}$$

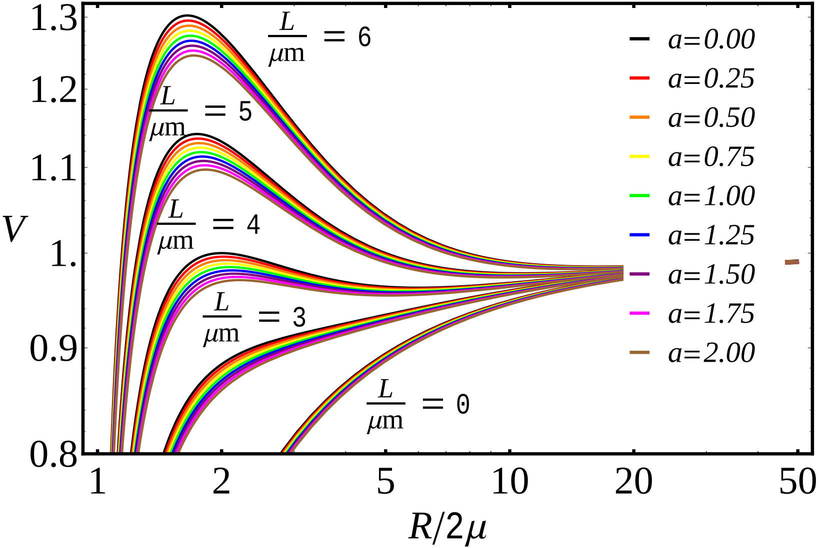
$R_0 > 2\mu$ and $R_- < 2\mu$.

After substitution of Eq. (2.43) to Eq. (2.29) and Eq. (2.32) we get

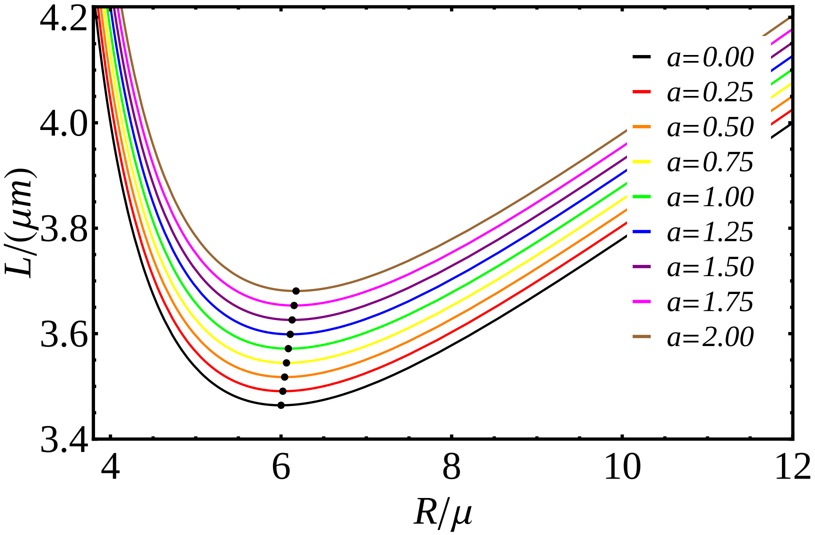
$$\frac{E^2}{m^2} = \frac{H^{-a}(R - 2\mu)^2 \left[2R - P(a - 2) \right]}{2\Delta_0}. \tag{2.45}$$

The distinctions between the curves for different values of a are larger for larger values of Q/μ.

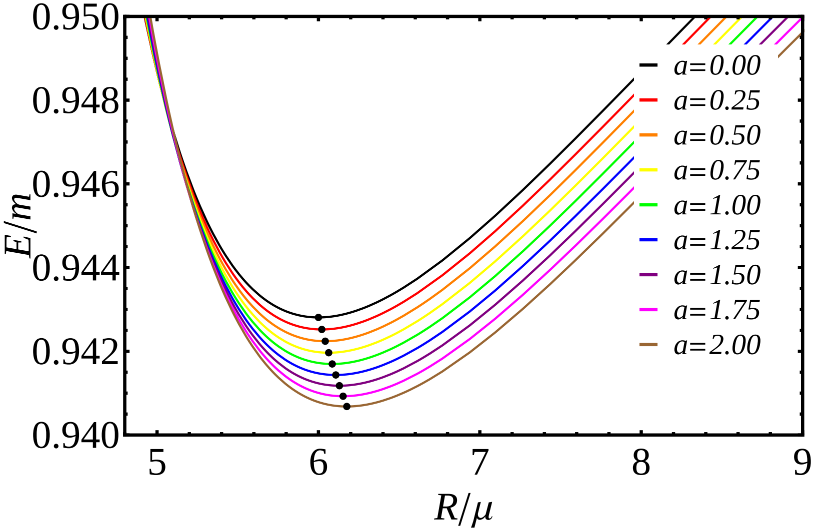
Effective potential for massive particles ($Q/\mu = 0.6$)



Orbital angular momentum of massive test particle



Energy of massive test particle ($Q/\mu = 0.6$)



Equations (2.43) (for L^2) and (2.45) (for E^2) reveal that for timelike geodesics, motion occurs only when $\Delta_0 > 0$, or, if

$$R > R_0. \tag{2.46}$$

It follows from Eq. (2.43) and Eq. (2.45) that

$$\frac{1}{m^2}\frac{dL^2}{dR} = \frac{R^2(1+P/R)^a}{\Delta_1^2}W, \tag{2.47}$$

$$\frac{1}{m^2}\frac{dE^2}{dR} = \frac{(1-2\mu/R)(1+P/R)^{-a}}{\Delta_1^2}W, \tag{2.48}$$

with

$$\Delta_1 = \frac{2}{R(R+P)(R-2\mu)}\Delta_0, \tag{2.49}$$

and

$$W = \frac{(2\mu)^5}{R^3(R+P)^3(R-2\mu)^2}F. \tag{2.50}$$

Master polynomial

$$
\begin{aligned}
F = F(x) = (2ap+2)x^4 + (6a(1-a)p^2 + (6-12a)p - 6)x^3 \\
+p(2(a-2)(a-1)ap^2 + 3(5a^2 - 9a + 2)p + 12a - 18)x^2 \\
-p^2((a-2)(4a^2 - 7a + 1)p + 9a^2 - 25a + 18)x \\
+(a-2)(a-1)(2a-3)p^3. \qquad (2.51)
\end{aligned}
$$

Here $x = R/(2\mu)$, $p = P/(2\mu)$.

Innermost stable circular orbits (ISCO)

ISCO is a crucial concept in the study of objects moving around BHs, e.g. accretion disks.
It may be verified that

$$\frac{\partial^2 V^2}{\partial R^2} = 2V\frac{\partial^2 V}{\partial R^2} = \frac{(1-2\mu/R)(1+P/R)^{-a}}{\Delta_1}W. \qquad (2.52)$$

Here $\Delta_1 > 0$, since $\Delta > 0$.
ISCO obeys

$$\frac{\partial^2 V}{\partial R^2} = 0, \qquad (2.53)$$

or, equivalently, $\frac{dL^2}{dR} = 0$ or $\frac{dE}{dR} = 0$. This implies $W = 0$ and we get the **master equation** for $x = x_{isco} = R_{isco}/(2\mu)$:

$$F(x) = 0. \qquad (2.54)$$

Solution to master equation

Proposition. *Let us consider 4-th order polynomial master equation* $F(x) = 0$*, where* $0 < a < 2$ *and* $p > 0$*. Then, the master equation* $F(x) = 0$ *has one and only one solution* $x_* = x_*(a, p)$ *which satisfies*

$$x_* > 1; \tag{2.55}$$

moreover, this solution obeys the bounds

$$x_* > x_0 \equiv \frac{1}{2}[(a-1)p + 3/2 + \sqrt{d}] > 1, \tag{2.56}$$

where $d = (1-a)^2p^2 + (3-a)p + 9/4 > 0$.

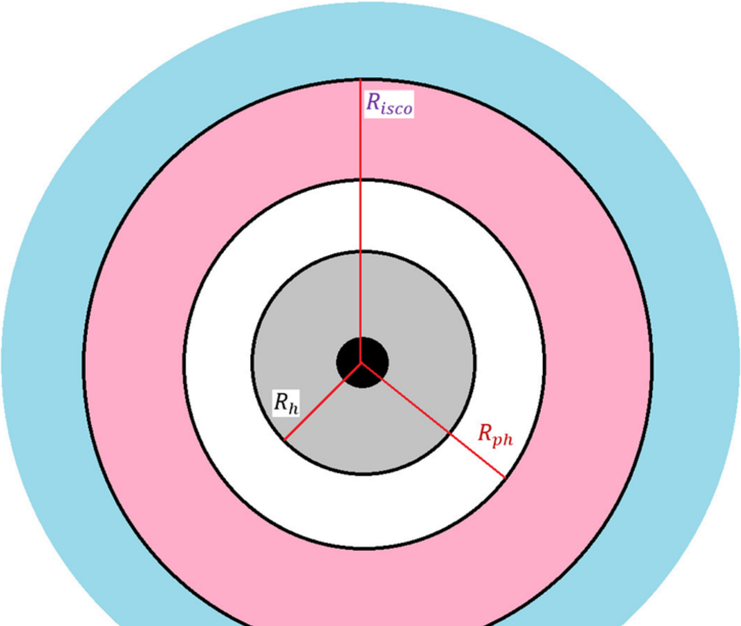
Key proposition on circular geodesics ($k = 1$)

The Proposition is proved in [BTIMNU, 2025]. It tells us about existence and uniqueness of $R_{isco} > 2\mu$, which also obeys the inequality $R_{isco} > R_0$ (R_0 is the radius of photon sphere).

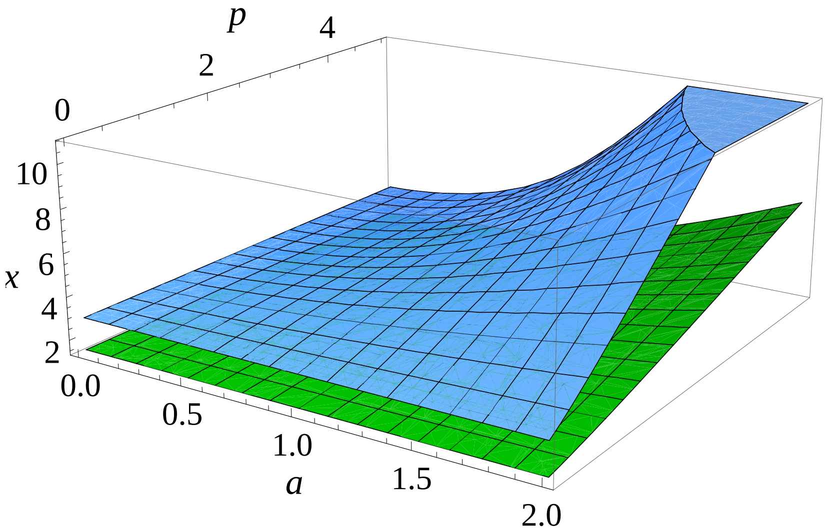
Theorem. *The circular time-like geodesics for massive neutral point-like particles are described by arbitrary radius $R > R_0$, where R_0 is the radius of photon sphere. The solutions with $R > R_{isco} = 2\mu x_{isco}$ are stable ($\frac{\partial^2 V}{\partial R^2} > 0$), while solutions with $R_0 < R < R_{isco}$ are unstable ($\frac{\partial^2 V}{\partial R^2} < 0$).*

The analytical solution for x_{isco} is presented in Appendix.

Figure for stable and unstable zones



3D plot of x_{ISCO} and x_0 as functions of a and p



The limit $p \to +\infty$ for $1 < a < 2$

For $1 < a < 2$ we have the following asymptotic relation

$$x_{isco}(a, p) \sim h(a)p, \tag{2.57}$$

as $p \to +\infty$, where

$$h(a) = (3/2)(a - 1) + (1/2)\sqrt{(a-1)(5a-1)} > 0. \tag{2.58}$$

For $1 < a < 2$ the relation for radius of photonic sphere (2.56) implies

$$x_0(a, p) \sim (a - 1)p, \tag{2.59}$$

as $p \to +\infty$. Here $x_0 < x_{isco}$.

The limit $p \to +\infty$, $0 < a < 1$

For $0 < a < 1$ we have asymptotical relation

$$x_{isco}(a, p) \to x_\infty(a), \tag{2.60}$$

as $p \to +\infty$, where

$$x_\infty(a) = \frac{\sqrt{-7a^2 + 10a + 1} - 4a^2 + 7a - 1}{4a(1-a)} > 1. \tag{2.61}$$

For $0 < a < 1$ the relation for radius of photonic sphere (2.56) implies

$$x_0(a, p) \to \frac{3 - 2a}{2(1-a)}, \tag{2.62}$$

as $p \to +\infty$. Here $x_0 < x_{isco}$.

R_{ISCO} for fixed μ and $P \to +\infty$

Let us fix the horizon radius $R_h = 2\mu$. Then, in the limit $P \to +\infty$, when $M \to +\infty$, $|Q| \to +\infty$ and

$$\frac{Q^2}{M^2} \to \frac{8}{a^2} \quad \text{(extremal BH limit)}. \tag{2.63}$$

we obtain

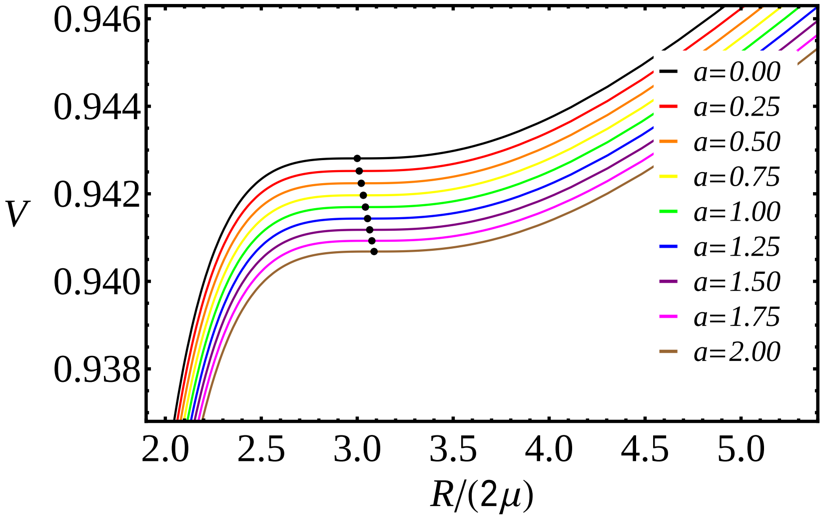
$$R_{ISCO} \sim h(a)P \sim h(a)|Q|/\sqrt{2}, \tag{2.64}$$

for $1 < a < 2$ and

$$R_{ISCO} \to x_\infty(a)2\mu, \tag{2.65}$$

for $0 < a < 1$.

The effective potential with $L = L_{ISCO}$ for different a



Here dots show the inflection points (corresponding to R_{ISCO}).

Efficiency of matter-to-radiation conversion

The parameter of the efficiency of matter-to-radiation conversion reads

$$\eta = [1 - \tilde{E}(R_{ISCO})] \times 100\%. \qquad (2.66)$$

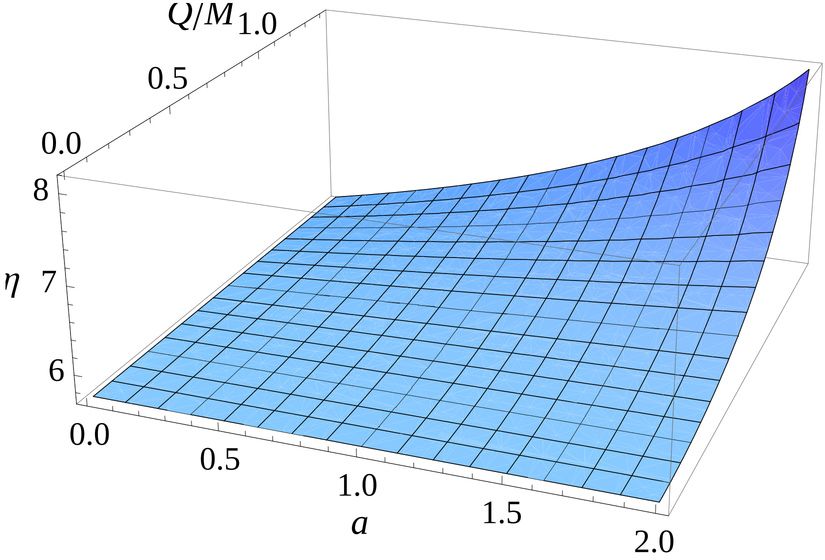
Here $\tilde{E}(R_{ISCO}) = E(R_{ISCO})/m$ is given by relation (2.45) and $R_{ISCO} = 2\mu x_4$, where $x_4 = x_4(a,p) > x_0 > 1$ is solution to master equation.

It is shown numerically that for $0 < a < 2$

$$\eta(0,0) < \eta = \eta(a,p) < \eta(2,p) < \eta(2,p=+\infty). \qquad (2.67)$$

where $\eta(0,0) \approx 5.72\%$ (Schwarzschild case) [Misner, Thorne, Wheeler] and $\eta(2,p=+\infty) \approx 8.14\%$ (extremal Reissner–Nordström case) [Bokhari,Rayimbaev,Ahmedov,2020].

Efficiency as a function of Q/M and a



Examples

For selected values of $a = 0, 1, 2$, we overview the following expressions for x_{isco}.

For the Schwarzschild BH ($a = 0$), the innermost stable circular orbit (ISCO) radius is [Kaplan, 1949]:

$$x_{isco}(a = +0) = 3. \tag{3.1}$$

The dimensionless radius of photon sphere reads

$$x_0(a = +0) = 3/2. \tag{3.2}$$

For $a = 1$, which corresponds to the Sen BH solution [Sen, 1992], the result is readily obtained [Boshkayev,Suliyeva,Iv.,Urazalina,2024]

$$x_{isco}(a = 1) = 1 + (1 + p)^{\frac{1}{3}} + (1 + p)^{\frac{2}{3}}. \tag{3.3}$$

For $a = 2$, we are led to the Reissner–Nordström BH case with the ISCO radius given by [Pugliese, Quevedo, Ruffini, 2011]

$$x_{isco}(a = 2) = 1 + p + X_2^{\frac{1}{3}} + \frac{1 + p + p^2}{X_2^{\frac{1}{3}}}, \tag{3.4}$$

where

$$X_2 = \frac{2 + p(1 + p)\left[7 + 4p(1 + p) + \sqrt{5 + 4p(1 + p)}\right]}{2(1 + 2p)}.$$

Extension to two independent charges for $a = 2$

Let us put

$$(\vec{\lambda}_1)^2 = (\vec{\lambda}_2)^2 = \vec{\lambda}_1 \vec{\lambda}_2 = \frac{1}{2}. \tag{3.5}$$

We get $a = 2$. The solution has a two parameter extension:

$$H^2 = \left(1 + \frac{P}{R}\right)^2 \longrightarrow H_1 H_2 = \left(1 + \frac{P_1}{R}\right)\left(1 + \frac{P_2}{R}\right). \tag{3.6}$$

We have two independent charge Q_1 and Q_2.
The master equation for radius of photon sphere x_0 is qubic one

$$x^3 + .. = 0. \tag{3.7}$$

[V.D.I, U. Kayumov, A.N. Malybayev, G.S. Nurbakova, Grav. Cosm., 2025].

Conclusions

- We have considered the solution for a dyonic-like dilatonic BH [MBI] described by parameters: $0 < a < 2$, $\mu > 0$, $P > 0$.
- We have explored circular null geodesics and related topics: BH shadows and QNM.
- We have investigated circular time-like geodesics .
- We have calculated analytically $R_{ISCO} > R_0$, where R_0 is the radius of photonic sphere. .
- The circular orbits of massive particles with radius R which obey $R > R_{ISCO}$ are shown to be stable while those satisfying $R_0 < R < R_{ISCO}$ are shown to be unstable.
- We have found an analytical relation for the efficiency of converting matter into radiation η and have proved (numerically) bounds on it: $\eta_{Schw} < \eta < \eta_{RN,ext}$.

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Thank you for your attention!