Black hole/Bose gase duality and entropic paradox

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Based on joint work with **I.Ya.Aref'eva** and **I.V.Volovich**

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Outline

- What is wrong with Schwarzschild Black Hole thermodynamics
- Duality
- BTZ Black Hole
- Black Brane/Bose gase duality
- Anistropic Lifschitz Black Brane /Bose gase duality
- Black String
- Conclusion
- Schwarzschild AdS black hole

Third Law of Thermodynamics

- Planck formulation. When temperature falls to absolute zero, the entropy tends to a universal constant (which can be taken to be zero)

$$S \to 0 \quad \text{as} \qquad T \to 0$$

  Entropy selected according to $S = 0$ at $T = 0$ is called absolute.
  M. Planck, Thermodynamik. 1911.
- Einstein statement. As the temperature falls to absolute zero, the entropy of any substance remains finite. this implies that S is finite at T $\to$ 0.
  A. Einstein, Annalen der Physik, 1907.
- "The third law of ordinary thermodynamics asserts that the entropy, S, of a system must go to zero (or a "universal constant") as its temperature, T, goes to zero."
  R.Wald arXiv:gr-qc/9704008

The third law of thermodynamics implies that as the temperature of an isolated system decreases, its energy and entropy—which measure disorder or motion—should also decrease. If entropy remained, it would imply motion without motion, creating as we call this problem an entropic paradox or a contradiction of the third law.

Schwarzschild Black Hole in $D = 4$

The metric of the Schwarzschild black hole in $D = 4$ spacetime is

$$ds^2 = -\left(1 - \frac{r_h}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{r_h}{r})} + r^2 d\mathbf{\Omega}_2^2, \tag{1}$$

with event horizon defined as, $r_h = 2G_4 M$, M is the mass of the black hole.
The Hawking temperature and the Bekenstein-Hawking entropy are

$$T = \frac{1}{8\pi G_4 M}, \qquad S = \frac{r_h^2}{4G_4} = \frac{1}{16\pi G_4 T^2}. \tag{2}$$

Problem with the 3LOT: The entropy goes to infinity as $T \to 0$ and we see a violation of the third law in Planck's formulation.
Also black hole explosion problem. For $M \to 0$ Hawking temperature goes as $T \to \infty$. Hawking '74, Black Hole Explosions?

Schwarzschild Black Hole in D

The D-dimensional Schwarzschild metric is given by:

$$ds^2 = \left(1 - \left(\frac{r_h}{r}\right)^{D-3}\right) d\tau^2 + \left(1 - \left(\frac{r_h}{r}\right)^{D-3}\right)^{-1} dr^2 + r^2 d\boldsymbol{\Omega}^2_{D-2}, \qquad (3)$$

where $d\boldsymbol{\Omega}^2_{D-2}$ represents the metric on the D-2-sphere . The horizon r_h is related with the ADM mass

$$M = \frac{r_h^{D-3}(D-2)\Omega_{D-2}}{16\pi G_D}, \qquad \Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)} \qquad (4)$$

Temperature and Entropy of the Schwarzschild black hole is

$$T = \frac{D-3}{4\pi r_h} = \frac{D-3}{4\pi}\left(\frac{(D-2)\Omega_{D-2}}{16\pi G_D M}\right)^{\frac{1}{D-3}}. \qquad (5)$$

$$S = \frac{\Omega_{D-2} r_h^{D-2}}{4\,G_D} = \frac{\Omega_{D-2}}{4\,G_D}\left(\frac{D-3}{4\pi T}\right)^{D-2} \qquad (6)$$

We see that the entropy increases when Hawking temperature T decreases, i.e the third law of thermodynamics is violated.

Bose-gase

Consider a Bose gas Free energy in a d-dimensional space

$$F_{BG} = \frac{1}{\beta} \sum_{\substack{k_i = 2\pi n_i / L \\ i = 1, \dots, d \\ n_i = 1, 2, \dots}} \ln \left(1 - e^{\beta\left(\mu - \sigma\, \varepsilon_\gamma(\vec{k})\right)}\right), \tag{7}$$

where $\varepsilon_\gamma(\vec{k})$ is the energy of quasi-particles:

$$\varepsilon_\gamma(\vec{k}) = k^\gamma \equiv \left(\vec{k}^2\right)^{\gamma/2}, \qquad \vec{k} = \{k_1, \dots, k_d\}, \tag{8}$$

and σ is a positive dimensional constant.Using spherical coordinates and integrating over spherical angles, we obtain for $\mu = 0$:

$$F_{BG} = \frac{\Omega_{d-1}}{\beta} \left(\frac{L}{2\pi}\right)^d \int_0^\infty \ln\left(1 - e^{-\beta\,\sigma\,k^\gamma}\right) k^{d-1}\, dk, \tag{9}$$

where $\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, and k^γ represents the exponent of the radial momenta as defined in (8).

Bose-gase

So we have

$$F_{BG} = -\left(\frac{L}{2\pi}\right)^d \frac{2\pi^{d/2}}{d\,\Gamma(d/2)} \left(\frac{1}{\beta}\right)^{\frac{d}{\gamma}+1} \left(\frac{1}{\sigma}\right)^{\frac{d}{\gamma}} \Gamma\left(\frac{d}{\gamma}+1\right) \zeta\left(\frac{d}{\gamma}+1\right). \tag{10}$$

The entropy is given by

$$\begin{aligned} S_{BG} &= -\frac{\partial F_{BG}(\beta,\gamma,\sigma,d,z)}{\partial T} = \beta^2 \frac{\partial F_{BG}}{\partial \beta} \\ &= \left(1+\frac{d}{\gamma}\right)\left(\frac{L}{2\pi}\right)^d \frac{2\pi^{d/2}}{d\,\Gamma(d/2)} \left(\frac{T}{\sigma}\right)^{\frac{d}{\gamma}} \Gamma\left(\frac{d}{\gamma}+1\right) \zeta\left(\frac{d}{\gamma}+1\right). \end{aligned} \tag{11}$$

Bose-gase Duality

This duality has been founded in I.Ya.Aref'eva, I.V.Volovich, 2304.04695.
By equating the entropy of a D-dimensional Schwarzschild black hole (6) with the entropy (11) of a d-dimensional Bose gas with energy (8), we obtain

$$S_{BH} = S_{BG}$$

$$\frac{\Omega_{D-2}}{4\,G_D}\left(\frac{D-3}{4\pi T}\right)^{D-2} = \left(1+\frac{d}{\gamma}\right)\left(\frac{L}{2\pi}\right)^{d}\frac{2\pi^{d/2}}{d\,\Gamma(d/2)}\left(\frac{T}{\sigma}\right)^{\frac{d}{\gamma}}\Gamma\left(\frac{d}{\gamma}+1\right)\zeta\left(\frac{d}{\gamma}+1\right)$$

$$T_{BH}^{-(D-2)} = T_{BG}^{\frac{d}{\gamma}} \tag{12}$$

which leads to

$$D-2 = -\frac{d}{\gamma}. \tag{13}$$

- This result implies that a positive $D-2$ corresponds to a negative d.
- Or that we have exotic Bose in positive d with negative $\gamma = -2$.

BTZ Black Hole

Consider 3 dimensional model of gravity with the action

$$S = \int d^3x\sqrt{-g}\,(R-\Lambda)\,, \tag{14}$$

where Λ is cosmological constant. The ansatz for the BTZ black hole in Schwarzschild coordinates (t, r, x) is given by M. Banados, C.Teitelboim, J.Zanelli, Phys. Rev. Lett. **69** (1992)

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + \frac{r^2}{L^2}\,dx^2. \tag{15}$$

Equation of motion system for this ansatz reduced for two equation

$$f'(r) + \Lambda r = 0, \tag{16}$$

$$f''(r) + \Lambda = 0. \tag{17}$$

By setting $\Lambda = -\frac{1}{l^2}$ one can find solution of this system is

$$f(r) = \frac{r^2}{l^2} - M, \tag{18}$$

here the constant l is the AdS length scale.

Third law of thermodynamics for BTZ

M represent mass of the black hole

$$M = \frac{r^2}{l^2}, \tag{19}$$

and horizon r_h is located at

$$r_h = l\sqrt{M}. \tag{20}$$

This black hole has the Hawking temperature

$$T_H = \frac{f'(r_h)}{4\pi} = \frac{r_h}{2\pi l^2}. \tag{21}$$

Or in mass terms is equal to

$$T_H = \frac{\sqrt{M}}{2\pi l}. \tag{22}$$

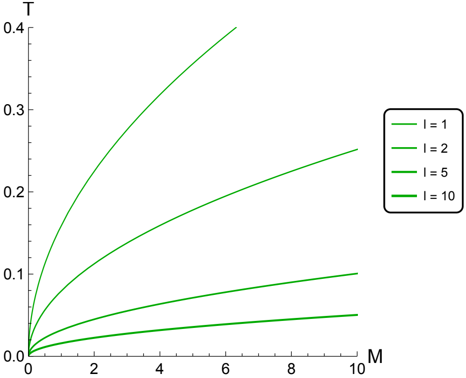
The entropy associated with the horizon has the form

$$S = 4\pi r_h, \tag{23}$$

One can express entropy in terms of temperature by expressing horizon value in terms of temperature $r_h = 2l^2\pi T_H$

$$S = 8l^2\pi^2 T_H. \tag{24}$$

Temperature dependence



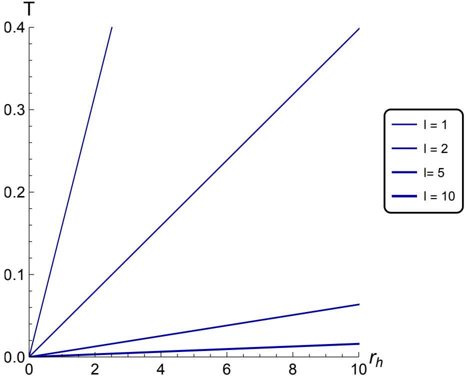


Figure: Plot shows Hawking temperature for BTZ blac hole of mass and horizon radious.

Entropy dependence

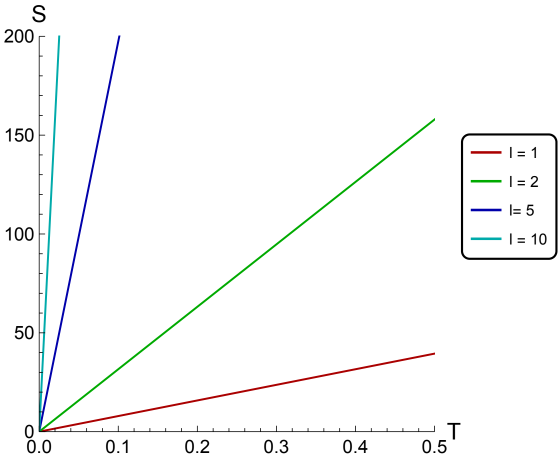


Figure: Entropy dependence from temperature with different length scale parameters.

Bose-gase Duality

Now let us compare thermodynamical properties of Bose gase in D-dimensions with BTZ black hole in $d = 3$ dimension. The entropy of Bose gase can be written as follows

$$S_{BG} = (1 + \frac{d}{\gamma}) \left(\frac{L}{2\pi}\right)^d \frac{2\pi^{d/2}}{d\Gamma(d/2)} \left(\frac{T}{\sigma}\right)^{\frac{d}{\gamma}} \Gamma\left(\frac{d}{\gamma} + 1\right) \zeta\left(\frac{d}{\gamma} + 1\right). \tag{25}$$

And BTZ entropy can be equal to entropy of Bose gase in the form

$$S_{BTZ} = 8l^2\pi^2 T_H = S_{BG}. \tag{26}$$

One can see that hold the equality for $\gamma = 2$,

$$T_H = T_{BG}^{\frac{d}{2}} \to d = 2. \tag{27}$$

By setting equality between entropys, we established that $d = 2$ Bose gase has the same behavior as entropy of $D = 3$ BTZ black hole.
Also from left part of this we have $\pi L^2/12\sigma = 8l^2\pi^2$. From which one can express cosmological length as

$$l = \frac{L}{4\sqrt{6\pi\sigma}} \tag{28}$$

Bose-gase Duality

Another case that can be considered is with $\gamma = 3$, which gives the relation to the dimensional parameter in ther form

$$T_H = T_{BG}^{\frac{d}{3}} \to d = 3. \tag{29}$$

Cosmological length in that case be found from

$$l = \frac{L^{3/2}}{12\pi\sqrt{\sigma}} \tag{30}$$

That means that we establish the duality between $d = 2$ and $d = 3$ Bose gase and $D = 3$ BTZ Black hole. This give us a statistical description of black hole entropy.

Poincare Black Brane in AdS

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} \left[R - \Lambda\right]. \tag{31}$$

The metric

$$ds^2 = \frac{L^2}{z^2} \left(-g(z)\, dt^2 + d\vec{x}^2_{D-2} + \frac{dz^2}{g(z)} \right), \tag{32}$$

with the blackening function given by

$$g(z) = 1 - \left(\frac{z}{z_h} \right)^{D-1}, \tag{33}$$

where z_h is the black hole horizon, defined by $g(z_h) = 0$, solves the equations of motion derived from the action (31), for

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2}, \tag{34}$$

where L is the AdS radius.

Entropy and Temperature

The Hawking temperature J.Maldacena arXiv:hep-th/0309246

$$T = \frac{D-1}{4\pi z_h}, \tag{35}$$

and the Bekenstein-Hawking entropy are

$$S = \frac{V_{D-2}}{4G_D}\frac{L^{D-2}}{z_h^{D-2}} = \frac{V_{D-2}}{4G_D}\left(\frac{4\pi LT}{D-1}\right)^{D-2}, \tag{36}$$

the volume V_{D-2} is determined by the area of the change of the coordinates $x_1, ...x_{D-2}$.
We see that $S \to 0$ as $T \to 0$, indicating that the third law of thermodynamics holds for the Poincare AdS black brane.

PAdS Black Brane/Bose gas duality

The entropy of a Bose gas in a d-dimensional space is $S_{BG} \sim T^{\frac{d}{\gamma}}$.
Comparing this with the entropy of the black brane $S_{BB} \sim T^{D-2}$, we see that both entropies have the same temperature dependence if

$$\frac{d}{\gamma} = D - 2. \tag{37}$$

This condition is satisfied in the special case $\gamma = 2$ where $D = d = 4$.

- I.Ya.Aref'eva, I.V.Volovich, D.S, arXiv 2411.01778 [hep-th]

Lifshitz solution

Consider the action in the form M.Taylor 0812.0530.

$$S = \frac{1}{16\pi G_D}\int d^D x\sqrt{-g}[R-\Lambda-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{\lambda\phi}F_{\mu\nu}F^{\mu\nu}], \tag{38}$$

The metric and the blackening function has the form

$$ds^2 = L^2[-r^{2\alpha}f(r)dt^2+\frac{dr^2}{r^2f(r)}+r^2\sum^{D-2}dx_i^2], \qquad f(r)=1-\frac{r_h^{\alpha+D-2}}{r^{\alpha+D-2}} \tag{39}$$

with matter fields are massles scalar and guage fields.

$$F_{rt} = q\, e^{-\lambda\phi}r^{\alpha-D-3}, \qquad \phi=\pm\sqrt{2(\alpha-1)(D-2)}\log r \tag{40}$$

solves EOM for action (38) for α, L and q related with λ and Λ as

$$\alpha = 1+\frac{2(D-2)}{\lambda^2}, \qquad L^2=-\frac{(\alpha+D-3)(\alpha+D-2)}{\Lambda}, \tag{41}$$

$$q^2 = 2L^2(\alpha-1)(\alpha+D-2). \tag{42}$$

We assume that $\alpha\geq 1$. Note that $\alpha=1$ corresponds to $\phi=0$ and reproduces plane brane solution (32).

Lifshitz Temperature and Entropy

The temperature is

$$T = \frac{(\alpha + D - 2)}{4\pi L}\, r_h^{\alpha}. \tag{43}$$

One can write the entropy as a function of temperature

$$S_{BH} = \frac{V_d}{4G_D}\,(L\, r_h)^{D-2} = \frac{V_d L^{D-2}}{4G_D}\left(\frac{4\pi}{\alpha + D - 2} L\, T\right)^{\frac{D-2}{\alpha}}, \tag{44}$$

here V_{D-2} is dimensionless.

Duality

Equalizing the entropy of the perfect Bose gas of quasi-particles and entropy of Lifshitz black brane given by (44) we get an equality

$$\left(\frac{L_{BG}}{2\pi^{1/2}}\right)^{d}\frac{\Gamma\left(\frac{d}{\gamma}+2\right)}{\Gamma(\frac{d}{2}+1)}\zeta\left(\frac{d}{\gamma}+1\right)\left(\frac{T}{\lambda}\right)^{\frac{d}{\gamma}}=\frac{V_{D-2}L^{D-2}}{4G_D}\left(\frac{4\pi L\,T}{\alpha+D-2}\right)^{\frac{D-2}{\alpha}} \tag{45}$$

We see that if we take (need requirement that $\gamma > 1$).

$$d=D-2,\qquad \gamma=\alpha \tag{46}$$

We can also suppose that d and D are related as

$$dn=D-2,\quad \text{or}\qquad d=(D-2)m, \tag{47}$$

where n and m are natural numbers. Since from (45) we get

$$\frac{d}{\gamma}=\frac{D-2}{\alpha}, \tag{48}$$

relations (**??**) and (47) give us

$$\alpha=n\gamma\quad\text{or}\quad m\alpha=\gamma. \tag{49}$$

The first solution in (49) allows us to abandon the requirement $\gamma > 1$.

Black String

The Einstein-Hilbert action for the four-dimensional uncharged black string is

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g}(R + 6\alpha^2), \tag{50}$$

EOM with negative cosmological constant is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 3\alpha^2 g_{\mu\nu}, \quad \alpha^2 = -\frac{\Lambda}{3} > 0. \tag{51}$$

The general solution in the cylindrically symmetric spacetime (t, r, ϕ, z) is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2 + \alpha^2 r^2 dz^2, \quad 0 \le \phi \le 2\pi, z < +\infty$$

$$f(r) = \left(\alpha^2 r^2 - \frac{4M}{\alpha r}\right), \quad r_+ = \frac{1}{\alpha} (4M)^{\frac{1}{3}} \tag{52}$$

- J. P. S. Lemos, Phys. Lett. B 353, 46 (1995).
- R. G. Cai, Y. Zhang, Phys. Rev. D 54 (1996) 4891
- J.Kumar,S.Upadhyay and H.Sudhanshua, Phys. Scr. 98 (2023) 095306

Thermodynamics of Black String

The Hawking temperature is

$$T = \left.\frac{f'(r)}{4\pi}\right|_{r=r_+} = \frac{3\alpha^2 r_+}{4\pi}. \tag{53}$$

The first law of thermodynamics leads to

$$S = \int \frac{dM}{T} = \frac{\pi\alpha r_+^2}{2}. \tag{54}$$

One can express entropy as function of temperature

$$S = \frac{2\pi T^2}{9\alpha^3} \tag{55}$$

Black String/Bose-gase Duality

Now let us compare thermodynamical properties of Bose-gase in D-dimensional with black string in

$$S_{BG} = (1 + \frac{d}{\gamma}) \left(\frac{L}{2\pi}\right)^d \frac{2\pi^{d/2}}{d\Gamma(d/2)} \left(\frac{T}{\sigma}\right)^{\frac{d}{\gamma}} \Gamma\left(\frac{d}{\gamma}+1\right) \zeta\left(\frac{d}{\gamma}+1\right) \quad (56)$$

$$S_{BS} = \frac{2\pi T^2}{9\alpha^3}$$

$$T^2 = T^{\frac{d}{\gamma}} \to \frac{d}{\gamma} = 2 \quad (57)$$

- So there is duality between $D = 4$ black string and $d = 4$ Bose-gase with $\gamma = 1$.
- And $d = 3$ Bose-gase with $\gamma = 3/2$
- I.V.Volovich, D.S., Black Hole Entropic Paradox, [In preparation].

Conclusion

- We found that Black Branes solution does not vioalate Third Law of Thermodynamics
- We establish duality between Bose-gase and Blach Branes
- We also found the duality between Lifshitz branes with exponent α in D-dimensional spacetime and Bose gases of quasi-particles with energy $(\vec{k}, \vec{k})^{\alpha/2}$ in $D-2$ spatial dimensions.
- Black String (With charged)
- BTZ Black Hole (with rotation)

The end.

Thank You For Your Attention!

Is there any questions ?

Schwarzschild-AdS Black Hole

The Einstein action with the present of cosmological constant Λ is

$$S = -\frac{1}{16\pi G}\int dx^D \sqrt{-g}\,[R + 2\Lambda] \tag{58}$$

The metric of a black hole in Anti-de Sitter (AdS) space in global coordinates for D-dimensional spacetime

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2} \tag{59}$$

where $d\Omega^2_{D-2}$ is the metric on a unit $(d-2)$-sphere, and the function $f(r)$ is given by:

$$f(r) = 1 - \frac{2M}{r^{D-3}} + \frac{r^2}{L^2} \tag{60}$$

With L AdS radius related to the cosmological constant Λ as

$$\Lambda = -\frac{(d-1)(d-2)}{2L^2} \tag{61}$$

Temperature of Schwarzschild-AdS

The Hawking Temperature is Belhaj, 1210.4617

$$T = \frac{1}{\beta} = \frac{1}{4\pi r_h}\left((d-3) + \frac{(d-1)r_h^2}{L^2}\right). \tag{62}$$

Which has the minimal temperature, from $\frac{\partial T}{\partial r_h} = 0$,

$$-\frac{1}{4\pi r_h^2}\left[(d-3) - \frac{(d-1)r_h^2}{L^2}\right] = 0 \tag{63}$$

$$r_0 = \frac{\sqrt{d-3}}{\sqrt{d-1}} L \tag{64}$$

we get minimal temperature

$$T_0 = \frac{\sqrt{d-3}\sqrt{d-1}}{2\pi L} \tag{65}$$

Entropy

The entropy is related with r_h as

$$S = \frac{A}{4} = \frac{\pi^{\frac{d-1}{2}} r_h^{d-2}}{2\Gamma(\frac{d-1}{2})} \tag{66}$$

From (62) we have the following relation between r_h and T

$$\begin{aligned} r_h &= \frac{2\pi L^2 T \pm \sqrt{4\pi^2 L^4 T^2 - L^2(d-1)(d-3)}}{(d-1)} && (67) \\ &= \frac{2\pi L^2 T}{(d-1)} \left(1 \pm \sqrt{1 - \frac{(d-1)(d-3)}{4\pi^2 L^2 T^2}}\right) && (68) \end{aligned}$$

Solution (68) has meaning only for

$$1 - \frac{(d-1)(d-3)}{4\pi^2 L^2 T^2} \geq 0 \Rightarrow \frac{(d-1)(d-3)}{4\pi^2 L^2 T^2} \leq 1, \tag{69}$$

i.e. $T \to 0$ is not admitted!

Temperature and Entropy for Schwarzschild-AdS in D=4

The Hawking Temperature is Hawking, Page '83

$$T = \frac{1}{4\pi r_h} + \frac{3r_h}{4\pi L^2} = \frac{1}{4\pi r_h}(1 + \frac{3r_h^2}{L^2}) \to T_0 = \frac{\sqrt{3}}{2\pi L} \tag{70}$$

The entropy is

$$S = \frac{\pi^{\frac{4-1}{2}} r_h^{4-2}}{2\Gamma(\frac{4-1}{2})} = \pi r_h^2 \tag{71}$$

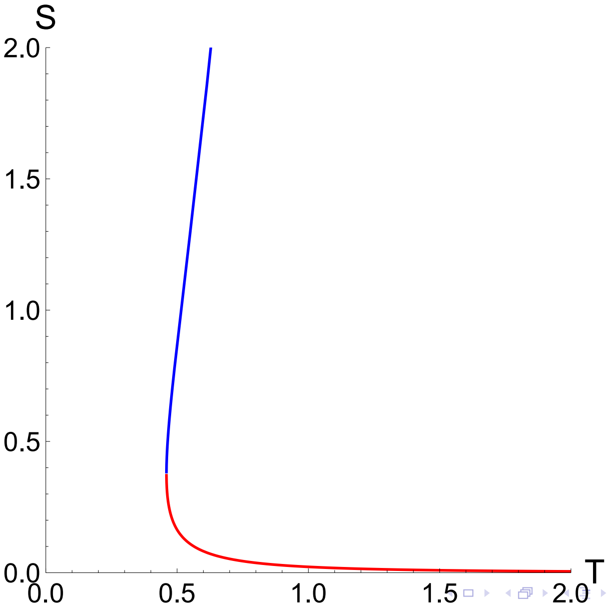
from relation

$$r_h = \frac{2\pi L^2 T}{3}\left(1 \pm \sqrt{1 - \frac{3}{4\pi^2 L^2 T^2}}\right) \to r_0 = \frac{1}{\sqrt{3}} L \tag{72}$$

Entropy is given by

$$S = \frac{1}{9}\pi L^4 T^2 \left(\sqrt{4\pi^2 - \frac{3}{L^2 T^2}} \pm 2\pi\right)^2 \to S_0 = \frac{\pi}{3} L^2 \tag{73}$$

Small and Large Black Holes



Temperature and Entropy for general metric

Consider the metric of the following form

$$ds^2 = B^2(z)\left(-g(z)N(z)dt^2 + \sum_i^{D-2} g_i dx_i^2 + \frac{dz^2}{K(z)g(z)}\right) \tag{74}$$

where $B(z), g(z), N(z), K(z)$ are smooth function defined on appropriate domain. With black hole horizon z_h is defined as a root of blackening function

$$g(z_h) = 0, \tag{75}$$

and also $N(z) > 0, K(z) > 0$.

$$T = \frac{1}{4\pi} g'(z_h)\sqrt{K(z_h)N(z_h)} \tag{76}$$

To calculate entropy we have assume horizon metric with fixed t, z.

$$S = \frac{V_{D-2}}{4G_D}\left(B(z_h)\right)^{D-2}\prod_{i=1}^{D-2}\sqrt{g_i(z_h)}, \tag{77}$$

Reissner-Nordström Black Hole

The metric of the Reissner-Nordström black hole in $D = 4$ spacetime is

$$ds^2 = -\left(1 - \frac{r_h}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r_h}{r} + \frac{Q^2}{r^2}} + r^2 d\Omega_2^2, \tag{78}$$

where $d\Omega_2^2$ is the metric on a 2-dimensional unit sphere and Q is the charge of the black hole. In this case, the parameters are $B^2(r) = 1$, $K(r) = N(r) = 1$, and $g(r) = \left(1 - \frac{r_h}{r} + \frac{Q^2}{r^2}\right)$.

The Hawking temperature and the Bekenstein-Hawking entropy are

$$T = \frac{r_h - Q^2/r_h}{4\pi r_h^2}, \qquad S = \frac{\pi r_h^2}{G_4}. \tag{79}$$

As the temperature T approaches zero, the entropy diverges, violating the third law of thermodynamics in Planck's formulation.

Kerr black hole

The metric of the Kerr black hole in $D = 4$ spacetime in ....coordinates is

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$

$$+\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2,$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Here M is the mass and $a = J/M$ is the specific angular momentum of the black hole.
The Hawking temperature and the Bekenstein-Hawking entropy are

$$T = \frac{r_h - M}{4\pi(r_h^2 + a^2)}, \qquad S = \frac{\pi(r_h^2 + a^2)}{G_4}. \tag{80}$$

As $T \to 0$, the entropy tends to infinity, violating the third law of thermodynamics in Planck's formulation.