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ON ELEMENTARY PARTICLE PHYSICS
MOSCOW STATE UNIVERSITY

Covariant reggeization in hadronic diffraction

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Plan

- ❑ Motivations
- ❑ Introduction
- ❑ Basics of the framework
- ❑ Calculations
- ❑ Further study

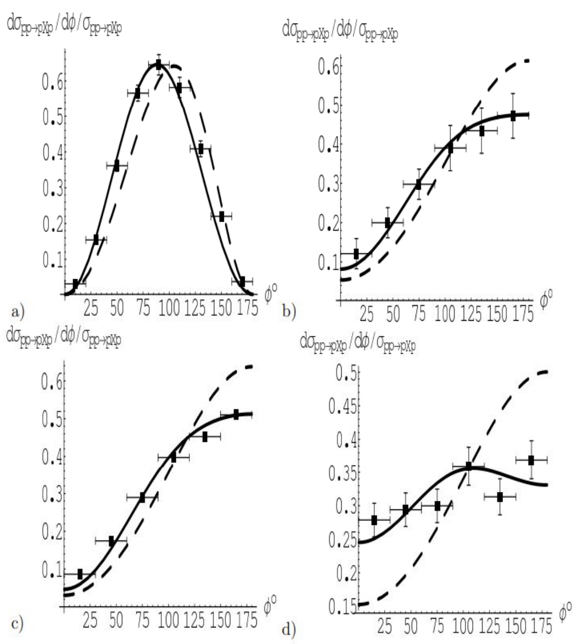
[V.A. Petrov, R.A. Ryutin, A.E. Sobol and J.-P. Guillaud, *Azimuthal Angular Distributions in EDDE as Spin-Parity Analyser and Glueball Filter for LHC*, JHEP06(2005)007]

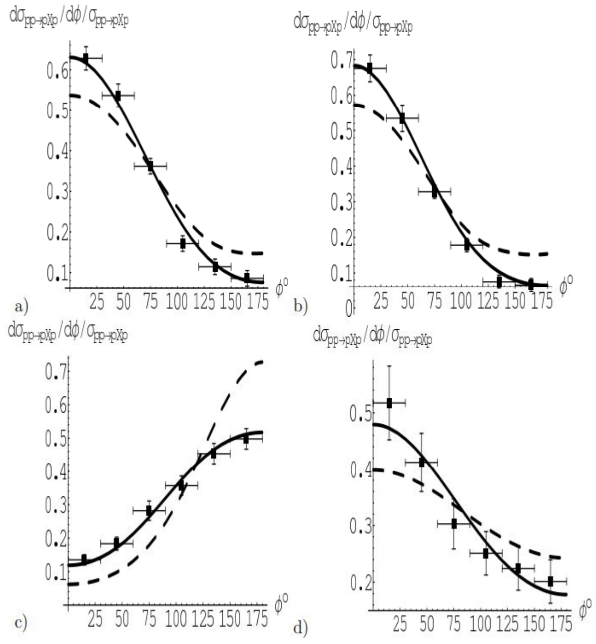
[R.A. Ryutin , *Visualizations of exclusive central diffraction*, Eur. Phys. J. C 74, 3162 (2014)]

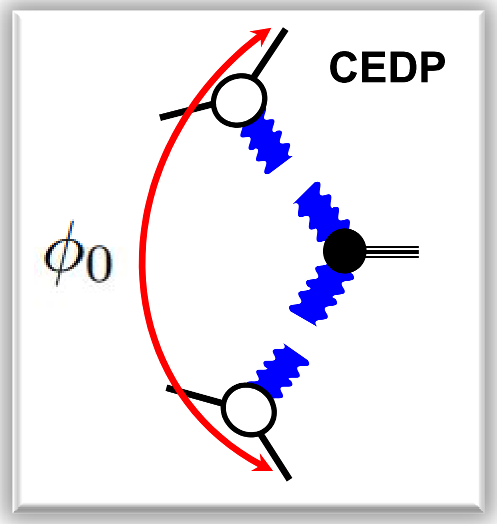
[V.A. Petrov and R.A. Ryutin, *Single and double diffractive dissociation and the problem of extraction of the proton–Pomeron cross-section*, Int. J. Mod. Phys. 31, No. 10, 1650049 (2016)]

[R. Ryutin, *Covariant reggeization framework for diffraction. Part I: Hadronic tensors in Minkovsky space-time of any dimension* arXiv:2507.16019 [hep-ph]]

Motivations: spin-parity analyzer







[A.B. Kaidalov, V.A. Khoze, A.D. Martin et al, ***Central exclusive diffractive production as a spin-parity analyser: from hadrons to Higgs. Eur. Phys. J. C*** **31, 387–396 (2003)]**

[V.A. Petrov, R.A. Ryutin, A.E. Sobol and J.-P. Guillaud, ***Azimuthal angular distributions in EDDE as spin-parity analyser and glueball filter for LHC*****, JHEP06(2005)007]**

Motivations:
Problems with conserved currents

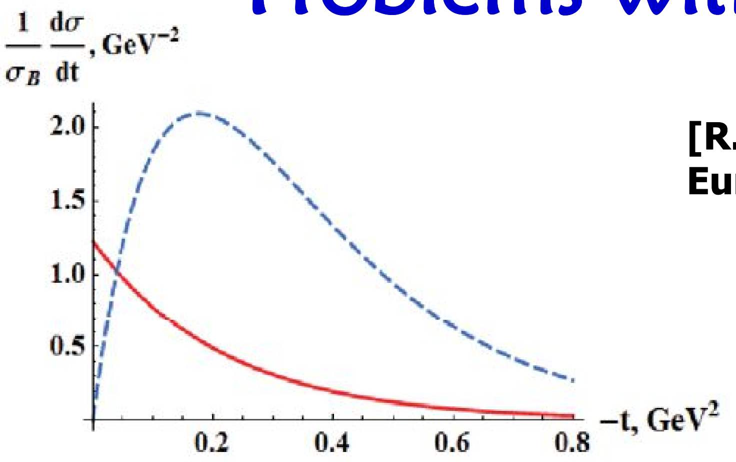


Fig. 2. The unitarization of the cross-section $|t|e^{-2B|t|}$ ($B \simeq$ 2.85 GeV^{-2}, $\sqrt{s} = 7$ TeV) corresponding to the amplitude (89) in the Appendix C. The dashed curve represents the "bare" term and the solid one represents the unitarized result. σ_B is the integrated "bare" cross-section. The zero at $t = 0$ disappears in the unitarized cross-section.

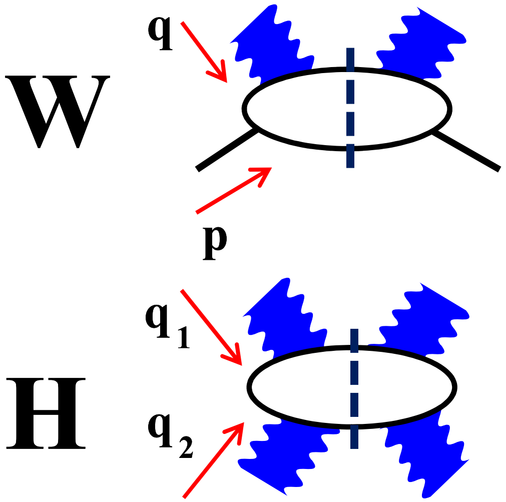
[R.A. Ryutin , *Visualizations of exclusive central diffraction*, Eur. Phys. J. C 74, 3162 (2014)]

"Non-Conserved" = Unitarization? Extra Dimension? Both?

[F.E. Close, G.A. Schuller, *Evidence that the pomeron transforms as a nonconserved vector current*, Phys. Lett. B 464, 279 (1999)]

[F.E. Close, G.A. Schuller, *Central production of mesons: Exotic states versus pomeron structure*, Phys. Lett. B 458, 127 (1999)]

Motivations: reggeon cross-sections



Hadron-Reggeon cross-section

Reggeon-Reggeon cross-section

Hadron-Hadron cross-section

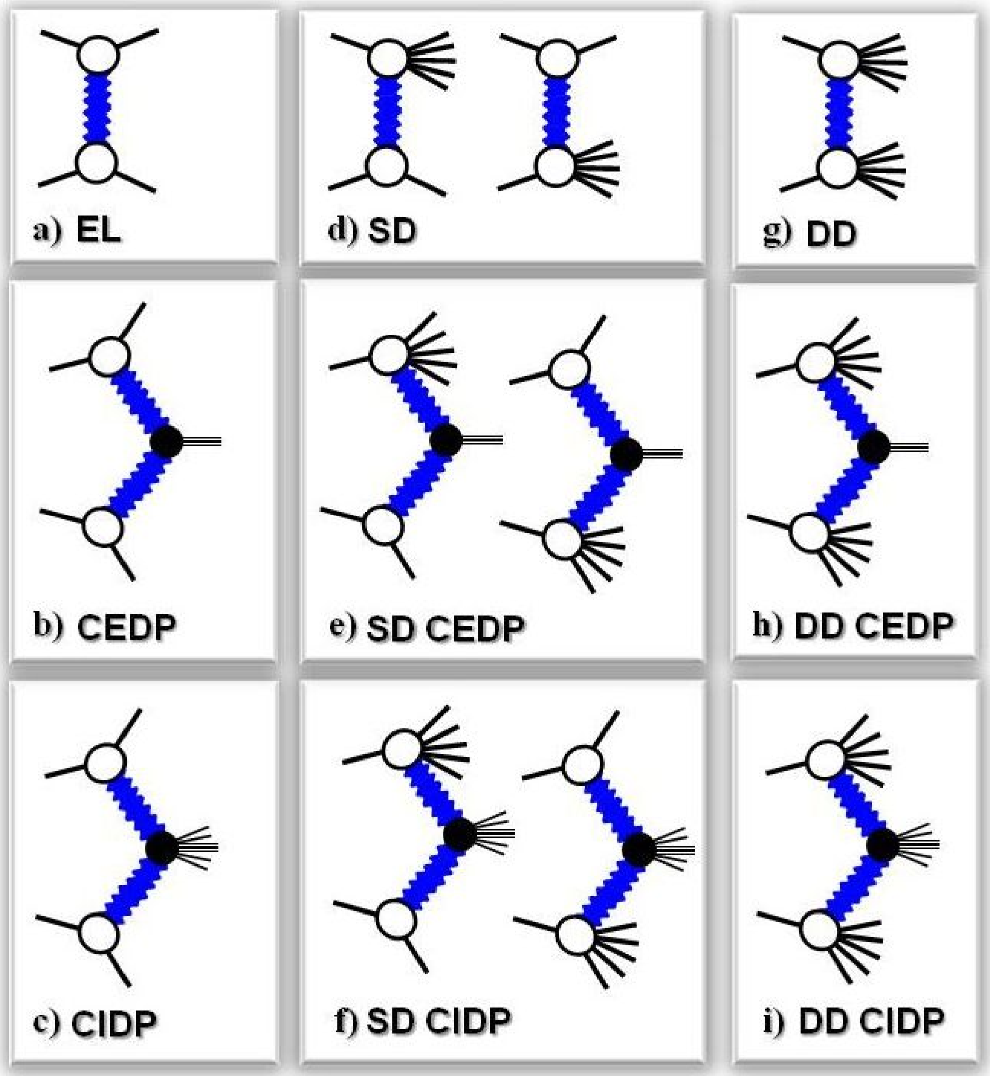
1 mb – 100 mb?

~10 – 100 mb

H-"Reggeon"?

What is the meaning of these values?

Introduction



Soft diffractive processes ("bare" amplitudes)

What is a "*REGGEON*"? "*POMERON*"?

(trajectory, intercept, residue, slope, ...)

What if the reggeon is a quantum field?

What is “REGGEON”?

- ❑ Gribov Reggeons
(in 2+1 …)
- ❑ BFKL (QCD) …
- ❑ Classical …
(10+ phenomenological models)

- ❑ Continuous & High Spin Particles
(Wigner, Fronsdal etc.)
- ❑ Hyper-Fields & Strings

+ Complex Momentum Plane

Basics

Current operator related to the hadronic spin-J Heisenberg field operator

$$\left(\Box + m_J^2\right) \Phi^{\mu_1 \dots \mu_J}(x) = \mathscr{J}^{\mu_1 \dots \mu_J}(x)$$

Rarita-Schwinger TST conditions

$$\partial_\mu \mathscr{J}^{\mu_1 \dots \mu \dots \mu_J} = 0 \, ;$$

$$\mathscr{J}^{\mu_1 \dots \mu_J} = \mathscr{J}^{(\mu_1 \dots \mu_J)} \, ;$$

$$g_{\mu_i \mu_k} \mathscr{J}^{\mu_1 \dots \mu_i \dots \mu_k \dots \mu_J} = 0$$

$$q_\lambda \mathscr{V}^{\mu_1 \dots \lambda \dots \mu_J} = 0 \, ;$$

$$\mathscr{V}^{\mu_1 \dots \mu_J} = \mathscr{V}^{(\mu_1 \dots \mu_J)} \, ;$$

$$g_{\mu_i \mu_k} \mathscr{V}^{\mu_1 \dots \mu_i \dots \mu_k \dots \mu_J} = 0$$

Basics

Scalar-Scalar-Spin-J vertex

$$\mathscr{V}^{\mu_1\ldots\mu_J}(p,q) =< p-q|\mathscr{I}^{\mu_1\ldots\mu_J}|p>$$

Spin-J Propagator

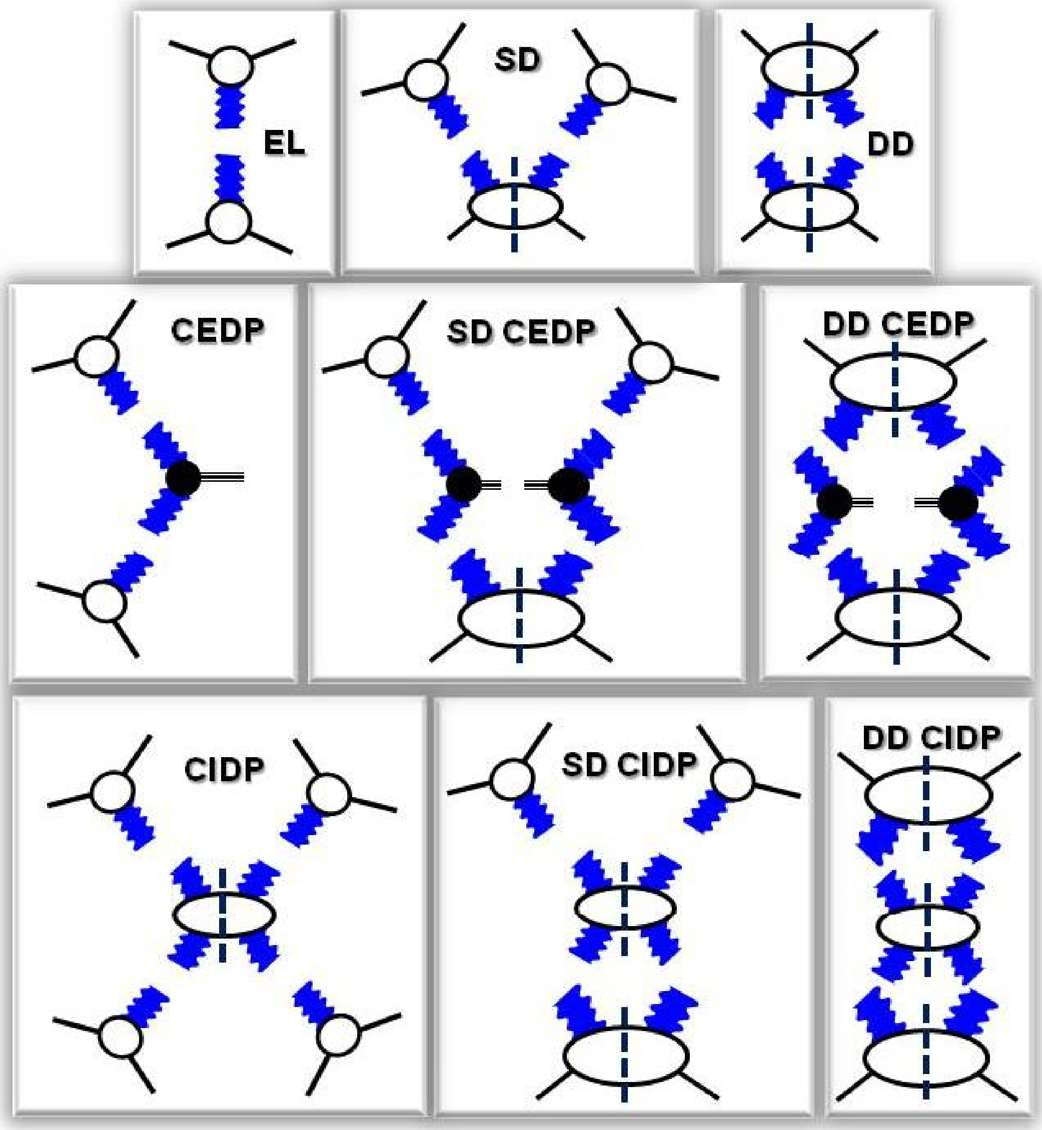
$$\mathscr{P}^{\mu_1\ldots\mu_J\nu_1\ldots\nu_J}(q) = \int d^4x\, e^{iq(x-y)} \langle 0|\, \Phi^{\mu_1\ldots\mu_J}(x)\Phi^{\nu_1\ldots\nu_J}(y)\, |0\rangle = \Pi^{\mu_1\ldots\mu_J,\nu_1\ldots\nu_J}(q)/(m^2(J)-q^2),$$

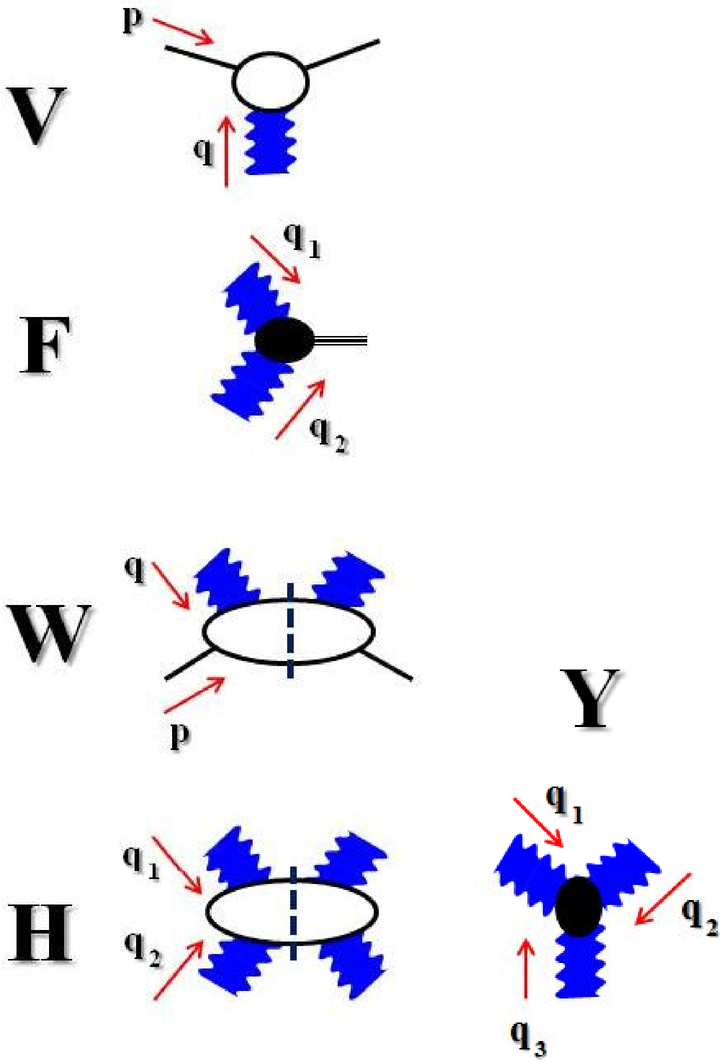
Pole at $m^2(J) - q^2 = 0,\ \text{i.e. } J = \alpha_{\mathbb{R}}(q^2)$

Reggeization prescription

$$\sum_J \frac{F^J}{(q^2-m^2)} \to \frac{\alpha'_{\mathbb{R}}}{2}\eta_{\mathbb{R}}(q^2)\Gamma(-\alpha_{\mathbb{R}}(q^2))F^{\alpha_{\mathbb{R}}(q^2)}$$

Diffractive Lego





Irreducible tensors

$$(\mu)_J \equiv (\mu_1\mu_2 \ldots \mu_J)$$

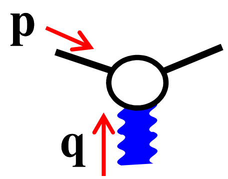
$$(1) \leftrightarrow (\mu)_{J_1}$$

$$(1') \leftrightarrow (\mu')_{J_{1'}}$$

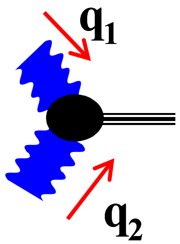
$$(2) \leftrightarrow (\nu)_{J_2}$$

$$(2') \leftrightarrow (\nu')_{J_{2'}}$$

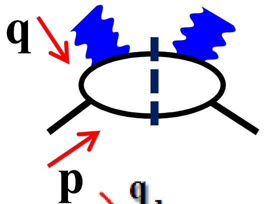
$$(3) \leftrightarrow (\rho)_{J_3}$$



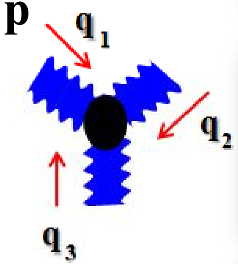
$$\mathcal{V}^J \equiv \mathcal{V}^J_{(1)}(p,q)$$



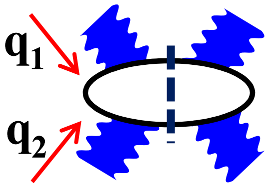
$$\mathcal{F}^{J_1,J_2} \equiv \mathcal{F}^{J_1,J_2}_{(1),(2)}(q_1,q_2)$$



$$\mathcal{W}^{J_1,J_{1'}} \equiv \mathcal{W}^{J_1,J_{1'}}_{(1),(1')}(p,q)$$



$$\mathcal{Y}^{\vec{J}} \equiv \mathcal{Y}^{\{J_1,J_2,J_3\}} \equiv \mathcal{Y}^{\vec{J}}_{(1),(2),(3)}(q_1,q_2)$$



$$\mathcal{H}^{\vec{J}} \equiv \mathcal{H}^{\{J_1,J_{1'},J_2,J_{2'}\}} \equiv \mathcal{H}^{\vec{J}}_{(1),(1'),(2),(2')}(q_1,q_2)$$

Transverse Symmetric tensors (V,W,F)

$$P_\alpha \equiv \frac{\left(p_\alpha - \frac{pq}{q^2} q_\alpha\right)}{\sqrt{p^2 - (pq)^2/q^2}}$$

$$P_{i\ \alpha} \equiv \frac{\left(p_{c\ \alpha} - \frac{p_c q_i}{q_i^2} q_{i\ \alpha}\right)}{\sqrt{M_c^2 - (p_c q_i)^2/q_i^2}}$$

$$Q_i^2 = -q_i^2 \equiv -t_i$$

$$\chi_m = \frac{Q_1 Q_2}{q_1 q_2}, \ \lambda_m = 1 - \chi_m^2$$

$$G_{(rr)} \equiv G_{(rr)\alpha_1\alpha_2} \equiv g_{\alpha_1\alpha_2} - \frac{q_{r\ \alpha_1} q_{r\ \alpha_2}}{q_r^2},$$

$$\alpha_{1,2} \in (r),$$

$$G_{(rr')} \equiv G_{(rr')\alpha\beta} \equiv g_{\alpha\beta} - \frac{q_{r\ \alpha} q_{r\ \beta}}{q_r^2},$$

$$\alpha \in (r) \,\&\, \beta \in (r'),$$

$$\hat{G}_{(rs)} \equiv \hat{G}_{(rs)\alpha\beta} \equiv g_{\alpha\beta} - \frac{q_{s\ \alpha} q_{r\ \beta}}{q_r q_s},$$

$$r \neq s, \ \alpha \in (r), \ \beta \in (s),$$

$$G_{\alpha\beta} q_{r\ \alpha} = 0, \ \hat{G}_{(rs)\alpha\beta} q_{r\ \alpha} = \hat{G}_{(rs)\alpha\beta} q_{s\ \beta} = 0$$

Transverse Symmetric tensors (V,W,F)

$$S_{n_1}^{V;J_1} \equiv \left(P_{(1)}^{J_1-2n_1} G_{(11)}^{n_1}\right)$$

$$S_{k',n_1 n_{1'}}^{W\{J_1,J_{1'}\}} \equiv \left(G_{(11')}^{k'} \prod_{i\in\Omega_W} P_{(i)}^{\bar{J}_i} G_{(ii)}^{n_i}\right)$$

$$\Omega_W = \{1,1'\},\ \bar{J}_i = J_i - 2n_i - k',$$

$$S_n^{P\ J} = S_{J-2n,nn}^{W\{J,J\}} \equiv \left(G_{(11')}^{J-2n} G_{(11)}^{n} G_{(1'1')}^{n}\right)$$

$$S_{k',n_1 n_2}^{F\{J_1,J_2\}} \equiv \left(\hat{G}_{(12)}^{k'} \prod_{i\in\Omega_F} P_{(i)}^{\bar{J}_i} G_{(ii)}^{n_i}\right)$$

$$\Omega_F = \{1,2\},\ \bar{J}_i = J_i - 2n_i - k',$$

Transverse Symmetric tensors (Y)

$$S^{Y\ \vec{J}}_{\vec{k}',\vec{n}} \equiv \left(\prod_{i \in \Omega_Y} P^{\bar{J}_i}_{(i)} G^{n_i}_{(ii)} \prod_{\substack{\forall r \neq s \\ r,s \in \Omega_Y}} \hat{G}^{k'_{rs}}_{(rs)} \right),$$

$$\Omega_Y = \{1,2,3\},\ \bar{J}_i = J_i - 2n_i - \sum_{\substack{r \neq i \\ r \in \Omega_Y}} k'_{ri},$$

Transverse Symmetric tensors (H)

$$S^{H\ \vec{J}}_{\vec{k}',\vec{n}} \equiv \left(G^{k'_{11'}}_{(11')} G^{k'_{22'}}_{(22')} \prod_{i\in\Omega_H} P^{\bar{J}_i}_{(i)} G^{n_i}_{(ii)} \prod_{\substack{\forall r\neq r',s \\ r,s\in\Omega_H}} \hat{G}^{k'_{rs}}_{(rs)} \right),$$

$$\Omega_H = \{1,1',2,2'\},\ \bar{J}_i = J_i - 2n_i - \sum_{\substack{r\neq i \\ r\in\Omega_H}} k'_{ri}$$

$$G_{(ii)\alpha\beta} = G_{(i'i')\alpha\beta} = G_{(ii')\alpha\beta},$$
$$G_{(ij)\alpha\beta} = G_{(ij')\alpha\beta} = G_{(i'j)\alpha\beta} = G_{(i'j')\alpha\beta},$$
$$i,j \in \{1,1',2,2'\},\ i \neq j.$$

Irreducible tensors (V)

$$\mathcal{V}^J(p,q) = \hat{v}_0(t) \sum_{n=0}^{[J/2]} v_n^J S_n^{V;J}$$

$$\begin{cases} v_{n-1}^J - 2\left(c^J + n - 1\right) v_n^J = 0, \ n > 0, \\ v_0^J = 1, \, c^J = -(J + (D-5)/2) \end{cases}$$

$$v_n^J = \frac{1}{2^n (c^J)_n}$$

Irreducible tensors (F)

$$\mathcal{F}^{J_1,J_2}(q_{(1)},q_{(2)}) = \sum_{k=0}^{\min(J_1,J_2)} \hat{f}_k^{J_1,J_2}(t_1,t_2)\bar{\mathcal{F}}_k^{J_1,J_2}$$

$$\bar{\mathcal{F}}_k^{J_1,J_2} = \sum_{k'=0}^{k} \sum_{n_{1.2}=0}^{[(J_{1,2}-k')/2]} f_{n_1 n_2}^{k'(k;J_1,J_2)} S_{k',\, n_1\, n_2}^{F;J_1,J_2},$$

$$\begin{cases} f^{k'}_{n_1-1\;n_2} - 2(c^{J_1}+n_1-1)f^{k'}_{n_1 n_2} + 2\chi_m \bar{J}_2 f^{k'+1}_{n_1-1\;n_2} + \\ +2n_2 f^{k'+2}_{n_1-1\;n_2-1} - \lambda_m \bar{J}_2\left(\bar{J}_2-1\right) f^{k'+2}_{n_1-1\;n_2} = 0, \\ \\ f^{k'}_{n_1\;n_2-1} - 2(c^{J_2}+n_2-1)f^{k'}_{n_1 n_2} + 2\chi_m \bar{J}_1 f^{k'+1}_{n_1\;n_2-1} + \\ +2n_1 f^{k'+2}_{n_1-1\;n_2-1} - \lambda_m \bar{J}_1\left(\bar{J}_1-1\right) f^{k'+2}_{n_1\;n_2-1} = 0, \\ \\ c^{J_i} = -(J_i + (D-5)/2),\; \bar{J}_i = \left(J_i - 2n_i - k'\right), \end{cases}$$

Irreducible tensors (F)

$$f_{n_1 n_2}^{k'(k;J_1,J_2)} =$$

$$= 2^{k-k'-n_1-n_2-1} \chi_m^{k-k'} \times$$

$$\left\{ \frac{n_2!\,(J_2-2n_2-k')!}{(c^{J_1})_{n_1}\chi_m^{2n_2}} \Lambda_{R\ k-k'\ n_1 n_2}^{J_1 J_2 k}(\hat{x}) + \right.$$

$$\left. \frac{n_1!\,(J_1-2n_1-k')!}{(c^{J_2})_{n_2}\chi_m^{2n_1}} \Lambda_{R\ k-k'\ n_2 n_1}^{J_2 J_1 k}(\hat{x}) \right\},$$

$$\mathcal{R}_N^L(\hat{x}) = \sum_{i=0}^{N} \mathbb{C}_N^{N-L+2i} \mathbb{C}_{N-L+2i}^{i} (\hat{x}/4)^i$$

$$\hat{x} = -\lambda_m/\chi_m^2,\ \mathcal{R}_N^L(0) = \mathbb{C}_N^L$$

$$\alpha_{l\ n}^{J_1 J_2 k} = \frac{(J_1-k+l)!}{(J_1-k)!} \frac{1}{(J_2-k+l-2n)!\,n!\,(c^{J_2})_n}$$

$$\Lambda_{R\ l\ n_1 n_2}^{J_1 J_2 k}(\hat{x}) = \sum_{i=0}^{n_2} \mathbb{C}_{n_1}^{n_2-i} \sum_{j=0}^{l} \alpha_{j\ i}^{J_1 J_2 k} \mathcal{R}_i^j(\hat{x}) \mathcal{R}_{n_1-n_2+i}^{l-2n_2+2i-j}(\hat{x})$$

Irreducible tensors (W)

$$\mathcal{W}^{J_1,J_{1'}}(p_{(1)},q_{(1)}) = \sum_{k=0}^{\min(J_1,J_{1'})} \hat{w}_k^{J_1,J_{1'}}(t_1)\bar{\mathcal{W}}_k^{J_1,J_{1'}}$$

$$\bar{\mathcal{W}}_k^{J_1,J_{1'}} = \sum_{k'=0}^{k} \sum_{n_{1,1'}=0}^{\left[(J_{1,1'}-k')/2\right]} w_{n_1 n_{1'}}^{k'(k;J_1,J_{1'})} S_{k',\, n_1\, n_2}^{W;J_1,J_{1'}}$$

$$\begin{cases} w^{k'}_{n_1-1\ n_{1'}} - 2(c^{J_1}+n_1-1)w^{k'}_{n_1 n_{1'}} + 2\bar{J}'_1 w^{k'+1}_{n_1-1\ n_{1'}} + \\ +2n_{1'}w^{k'+2}_{n_1-1\ n_{1'}-1} = 0, \\ \\ w^{k'}_{n_1\ n_{1'}-1} - 2(c^{J_{1'}}+n_{1'}-1)w^{k'}_{n_1 n_{1'}} + 2\bar{J}_1 w^{k'+1}_{n_1\ n_{1'}-1} + \\ +2n_1 w^{k'+2}_{n_1-1\ n_{1'}-1} = 0, \\ \\ c^{J_1} = -(J_1+(D-5)/2),\ \bar{J}_1 = \left(J_1-2n_1-k'\right), \\ c^{J_{1'}} = -(J_{1'}+(D-5)/2),\ \bar{J}'_1 = \left(J_{1'}-2n_{1'}-k'\right), \end{cases}$$

Irreducible tensors (W)

$$w_{n_1 n_{1'}}^{k'(k;J_1,J_{1'})} =$$

$$= 2^{k-k'-n_1-n_{1'}-1} \times$$

$$\left\{ \frac{n_{1'}!\,(J_{1'} - 2n_{1'} - k')!}{(c^{J_1})_{n_1}} \Lambda_{\mathbb{C}\ k-k'\ n_1 n_{1'}}^{J_1 J_{1'} k} + \right.$$

$$\left. \frac{n_1!\,(J_1 - 2n_1 - k')!}{(c^{J_{1'}})_{n_{1'}}} \Lambda_{\mathbb{C}\ k-k'\ n_{1'} n_1}^{J_{1'} J_1 k} \right\},$$

$$\mathcal{P}^J(q) = \sum_{n=0}^{[J/2]} \frac{n!}{(c^J)_n} S_n^{P\ J}$$

$$\Lambda_{C\ l\ n_1 n_2}^{J_1 J_2 k} = \lim_{\hat{x}\to 0} \Lambda_{R\ l\ n_1 n_2}^{J_1 J_2 k}(\hat{x}) =$$

$$= \sum_{i=0}^{n_2} \mathbb{C}_{n_1}^{n_2-i} \sum_{j=0}^{l} \alpha_{j\ i}^{J_1 J_2 k} \mathbb{C}_i^j \mathbb{C}_{n_1-n_2+i}^{l-2n_2+2i-j}$$

General multivariate recurrence

$$b_n^{\vec{\kappa}'} = \left(\gamma + \sum_{i=1}^{L} \omega_i \hat{O}^{\vec{\Delta}_i}\right) b_{n-1}^{\vec{\kappa}'}$$

$$b_0^{\vec{\kappa}'} = B^{\vec{\kappa}'}$$

$$\vec{\Delta}_i = \{\Delta_{i,1}, \Delta_{i,2}, \ldots \Delta_{i,N}\}$$

$$(\mathcal{M}_\Delta)_{ij} \equiv \left(\vec{\Delta}_i\right)_j$$

$$\vec{\kappa} = \{k_1, k_2, \ldots, k_N\}$$

$$\vec{\kappa}' + \vec{\Delta} = \{k_1' + \Delta_1, k_2' + \Delta_2, \ldots, k_N' + \Delta_N\}$$

$$b_n^{\vec{\kappa}'} = \sum_{m=0}^{n} \frac{n!}{(n-m)!} \gamma^{n-m} \sum_{\substack{\sum_{j=1}^{L} m_j = m \\ m_i \geq 0}} \prod_{i=1}^{L} \frac{\omega_i^{m_i}}{m_i!} B^{\vec{\kappa}' + \sum_{r=1}^{L} m_r \vec{\Delta}_r} \equiv \hat{\mathcal{S}}^{\,\vec{\omega},\,\mathcal{M}_\Delta}\left[B^{\vec{\kappa}'}\right]$$

$$\hat{\mathcal{S}}^{\,\vec{\omega},\,\mathcal{M}_\Delta}_{a_1,\ldots,a_{N'}}\left[B^{\vec{\kappa}'}\right] = \hat{\mathcal{S}}^{\,\vec{\omega},\,\mathcal{M}_\Delta}\left[B^{\vec{\kappa}'}\right]\Big|_{m_{a_1} = \ldots = m_{a_{N'}} = 0}$$

Irreducible tensors (Y)

$$\mathcal{Y}^{\vec{J}}(q_1,q_2)=\sum_{\bar{\Omega}^Y_{\vec{k}}}\hat{y}^{\vec{J}}_{\vec{k}}(t_1,t_2)\bar{\mathcal{Y}}^{\vec{J}}_{\vec{k}},$$

$$\bar{\mathcal{Y}}^{\vec{J}}_{\vec{k}}=\sum_{\tilde{\Omega}^Y_{\vec{k}',\vec{n}}} y_{\vec{n}}^{\vec{k}'(\vec{k};\vec{J})} S^{Y;\vec{J}}_{\vec{k}',\vec{n}},$$

$$\tilde{\Omega}^Y_{\vec{k}',\vec{n}}$$

$$\begin{cases} \bar{J}_r\geq 0,\ r\in\Omega_Y \\ \sum\limits_{\substack{s\in\Omega_Y\\ s\neq r}} k'_{rs}\leq \sum\limits_{\substack{s\in\Omega_Y\\ s\neq r}} k_{rs} \\ n_r,\ k'_{rs}\in\mathbb{Z}_+ \end{cases}$$

$$\bar{\Omega}^Y_{\vec{k}}$$

$$\begin{cases} 0\leq k_{12}+k_{13}\leq J_1 \\ 0\leq k_{12}+k_{23}\leq J_2 \\ 0\leq k_{13}+k_{23}\leq J_3 \\ k_{rs}\in\mathbb{Z}_+, \end{cases}$$

Irreducible tensors (Y)

$$\hat{\mathcal{A}}_i^Y = \hat{O}_{n_i}^{-1} \times$$

$$\times \left(\bar{\kappa}_i + \left[2k'_{rs} \hat{O}^{+1}_{k'_{ir}} \hat{O}^{+1}_{k'_{is}} \hat{O}^{-1}_{k'_{rs}} + 2\hat{\lambda}^i_{rs} \bar{J}_r \bar{J}_s \hat{O}^{+1}_{k'_{ir}} \hat{O}^{+1}_{k'_{is}} \right] + \right.$$

$$\left. + \sum_{\substack{j \neq i \\ j \in \Omega_Y}} \left[2\bar{\chi}_{ij} \bar{J}_j \hat{O}^{+1}_{k'_{ij}} + 2n_j \hat{O}^{+2}_{k'_{ij}} \hat{O}^{-1}_{n_j} - \tilde{\lambda}^j_{ij} \bar{J}_j \left(\bar{J}_j - 1 \right) \hat{O}^{+2}_{k'_{ij}} \right] \right)$$

$$r, s, i, j \in \Omega_Y;\ i \neq r \neq s.$$

$$y^{\vec{k}'}_{\vec{n}} = \frac{1}{2(c^{J_i} + n_i - 1)} \hat{\mathcal{A}}_i^Y y^{\vec{k}'}_{\vec{n}}$$

Solution (Y)

Change of variables and initial condition

$$y_{\vec{n}}^{\vec{k}'} \equiv y_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})} = \frac{2^{\sum\limits_{\substack{r,s\in\Omega_Y\\ r\neq s}}(k_{rs}-k'_{rs})-\sum\limits_{r\in\Omega_Y} n_r}}{\prod\limits_{r\in\Omega_Y}\left(c^{J_r}\right)_{n_r}}\mathbf{y}_{\vec{n}}^{\vec{k}'},$$

$$\mathbf{y}_{\vec{n}}^{\vec{k}} = 1,\ \mathbf{y}_{\vec{0}}^{\vec{k}'} = \delta_{\vec{k}'\vec{k}} \equiv \prod_{\substack{r,s\in\Omega_Y\\ r\neq s}} \delta_{k'_{rs}k_{rs}}.$$

$$\mathbf{y}_{\vec{n}}^{\vec{k}'} = \mathbf{y}_{i;\,\vec{n}}^{\vec{k}'}\mathbf{A}_{i;\,\vec{n}}^{\vec{k}'},$$

$$\mathbf{A}_{i;\,\vec{n}}^{\vec{k}'} = \prod_{\substack{r\neq i\\ r\in\Omega_Y}} \bar{J}_r!n_r!\left(c^{J_r}\right)_{n_r}\cdot k'_{\hat{r}\hat{s}}!,$$

$$i,\hat{r},\hat{s}\in\Omega_Y,\ i\neq\hat{r}\neq\hat{s}.$$

$$\vec{\kappa}_i^{\prime Y} = \{k'_{ir},k'_{is},k'_{rs},n_r,n_s\},$$
$$\vec{\kappa}_i^{Y} = \{k_{ir},k_{is},k_{rs},n_r,n_s\},$$
$$i,r,s\in\Omega_Y,\ i\neq r\neq s.$$

Solution (Y)

Parameters for the general recurrence

$$\vec{\omega}_i^Y = \{1, \hat{\lambda}^i_{rs}/2, \bar{\chi}_{ir}, -\tilde{\lambda}^j_{ir}/4, \bar{\chi}_{is}, -\tilde{\lambda}^j_{is}/4, 1, 1\},$$

$$\mathcal{M}^Y_\Delta = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 \end{pmatrix}$$

Solution (Y)

$$\mathbf{y}_{\vec{n}}^{\vec{k}'} = \frac{1}{3!} \sum_{\substack{\{i,r,s\} \\ \in P\{1,2,3\}}} \mathbf{A}_i \, \hat{\mathcal{S}} \left[\left(\frac{\mathbf{A}_r}{\mathbf{A}_i} \right)_{n_{r,s} \neq 0}^{n_i = 0} \hat{\mathcal{S}}_{a_{is}} \left[\left(\frac{\mathbf{A}_s}{\mathbf{A}_r} \right)_{n_s \neq 0}^{n_{i,r}=0} \hat{\mathcal{S}}_{4,5} \left[\frac{\delta_{\vec{k}'\vec{k}}}{\mathbf{A}^{\vec{k}}_{s;\vec{0}}} \right] \right] \right]$$

$$\hat{\mathcal{S}}_{\dots} \equiv \hat{\mathcal{S}}_{\dots}^{\vec{\omega}_i^Y,\, \mathcal{M}_\Delta^Y} \qquad \mathbf{A}_i \equiv \mathbf{A}_{i;\,\vec{n}}^{\vec{k}'} \qquad a_{is} \equiv \begin{cases} 4 & i < s \\ 5 & i > s \end{cases}$$

$$y_{\vec{n}}^{\vec{k}'} \equiv y_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})} = \frac{2^{\sum\limits_{\substack{r,s \in \Omega_Y \\ r \neq s}} (k_{rs} - k'_{rs}) - \sum\limits_{r \in \Omega_Y} n_r}}{\prod\limits_{r \in \Omega_Y} \left(c^{J_r} \right)_{n_r}} \mathbf{y}_{\vec{n}}^{\vec{k}'}$$

Irreducible tensors (H)

$$\mathcal{H}^{\vec{J}}(q_1,q_2)=\sum_{\bar{\Omega}^H_{\vec{k}}}\hat{h}^{\vec{J}}_{\vec{k}}(t_1,t_2)\bar{\mathcal{H}}^{\vec{J}}_{\vec{k}},$$

$$\bar{\mathcal{H}}^{\vec{J}}_{\vec{k}}=\sum_{\tilde{\Omega}^H_{\vec{k}',\vec{n}}} h_{\vec{n}}^{\vec{k}'(\vec{k};\vec{J})} S^{H;\vec{J}}_{\vec{k}',\vec{n}},$$

$$\tilde{\Omega}^H_{\vec{k}',\vec{n}}$$

$$\begin{cases} \bar{J}_r\geq 0,\ r\in\Omega_H \\ \sum\limits_{\substack{s\in\Omega_H\\ s\neq r}} k'_{rs}\leq \sum\limits_{\substack{s\in\Omega_H\\ s\neq r}} k_{rs} \\ n_r,\ k'_{rs}\in\mathbb{Z}_+ \end{cases}$$

$$\bar{\Omega}^H_{\vec{k}}$$

$$\begin{cases} 0\leq k_{11'}+k_{12}+k_{12'}\leq J_1 \\ 0\leq k_{11'}+k_{1'2}+k_{1'2'}\leq J_{1'} \\ 0\leq k_{22'}+k_{12}+k_{1'2}\leq J_2 \\ 0\leq k_{22'}+k_{12'}+k_{1'2'}\leq J_{2'} \\ k_{rs}\in\mathbb{Z}_+, \end{cases}$$

Irreducible tensors (H)

$$\hat{\mathcal{A}}_i^H = \hat{O}_{n_i}^{-1}\Big(1 + 2\bar{J}_i\hat{O}_{k'_{ii'}}^{+1} + 2n_{i'}\hat{O}_{k'_{ii'}}^{+2}\hat{O}_{n_{i'}}^{-1} +$$

$$+2k'_{jj'}\hat{O}_{k'_{ij}}^{+1}\hat{O}_{k'_{ij'}}^{+1}\hat{O}_{k'_{jj'}}^{-1} - 2\lambda_m\bar{J}_j\bar{J}_{j'}\hat{O}_{k'_{ij}}^{+1}\hat{O}_{k'_{ij'}}^{+1} +$$

$$+\sum_{\substack{r\neq i,i'\\ r\in\Omega_H}}\Big[2\chi_m\bar{J}_r\hat{O}_{k'_{ir}}^{+1} + 2k'_{i'r}\hat{O}_{k'_{ii'}}^{+1}\hat{O}_{k'_{ir}}^{+1}\hat{O}_{k'_{i'r}}^{-1}$$

$$+2n_r\hat{O}_{k'_{ir}}^{+2}\hat{O}_{n_r}^{-1} - \lambda_m\bar{J}_r\left(\bar{J}_r - 1\right)\hat{O}_{k'_{ir}}^{+2}\Big]\Big),$$

$$r, i, i', j, j' \in \Omega_H;$$

$$\{j, j'\} \neq \{i, i'\},\ j \equiv \bar{i},\ j' \equiv \bar{i}'.$$

$$h_{\vec{n}}^{\vec{k}'} = \frac{1}{2(c^{J_i} + n_i - 1)}\hat{\mathcal{A}}_i^H h_{\vec{n}}^{\vec{k}'}$$

Solution (H)

Change of variables and initial condition

$$h_{\vec{n}}^{\vec{k}'} \equiv h_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})} = \frac{2^{\sum\limits_{\substack{r,s\in\Omega_H\\ r\neq s}}(k_{rs}-k'_{rs}) - \sum\limits_{r\in\Omega_H} n_r}}{\prod\limits_{r\in\Omega_H}\left(c^{J_r}\right)_{n_r}} \mathbf{h}_{\vec{n}}^{\vec{k}'},$$

$$\mathbf{h}_{\vec{n}}^{\vec{k}} = 1, \ \mathbf{h}_{\vec{0}}^{\vec{k}'} = \delta_{\vec{k}'\vec{k}} \equiv \prod_{\substack{r,s\in\Omega_H\\ r\neq s}} \delta_{k'_{rs}k_{rs}}.$$

$$\mathbf{h}_{\vec{n}}^{\vec{k}'} = \mathbf{h}_{i;\,\vec{n}}^{\vec{k}'} \mathbf{A}_{i;\,\vec{n}}^{\vec{k}'},$$

$$\mathbf{A}_{i;\,\vec{n}}^{\vec{k}'} = \prod_{\substack{r\neq i\\ r\in\Omega_H}} \bar{J}_r! n_r! \left(c^{J_r}\right)_{n_r} \prod_{\substack{r,s\in\Omega_H\\ r\neq s\neq i}} k'_{rs}!, \ i\in\Omega_H.$$

$$\vec{\kappa}_i'^{H} = \{k'_{ii'}, k'_{ij}, k'_{ij'}, k'_{i'j}, k'_{i'j'}, k'_{jj'}, n_{i'}, n_j, n_{j'}\},$$
$$\vec{\kappa}_i^{H} = \{k_{ii'}, k_{ij}, k_{ij'}, k_{i'j}, k_{i'j'}, k_{jj'}, n_{i'}, n_j, n_{j'}\}.$$

$$r, i, i', j, j' \in \Omega_H;$$
$$\{j, j'\} \neq \{i, i'\}, \ j \equiv \bar{i}, \ j' \equiv \bar{i}'.$$

Solution (H)

Parameters for the general recurrence

$$\vec{\omega}^H = \{1, 1, -\lambda_m/2, \chi_m, 1, -\lambda_m/4, \chi_m, 1, -\lambda_m/4, 1, 1, 1\};$$

$$\mathcal{M}^H_\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Solution (H)

$$\mathbf{h}_{\vec{n}}^{\vec{k}'} = \frac{1}{4!}\sum_{\substack{\{i,j,r,s\}\\ \in P\{1,1',2,2'\}}} \mathbf{A}_i\, \hat{\mathcal{S}}\left[\left(\frac{\mathbf{A}_j}{\mathbf{A}_i}\right)_{n_{j,r,s}\neq 0}^{n_i=0} \hat{\mathcal{S}}_{a_{rsj}}\left[\left(\frac{\mathbf{A}_s}{\mathbf{A}_j}\right)_{n_{r,s}\neq 0}^{n_{i,j}=0} \hat{\mathcal{S}}_{\{a_1,a_2\}_{rs}}\left[\left(\frac{\mathbf{A}_r}{\mathbf{A}_s}\right)_{n_r\neq 0}^{n_{i,j,s}=0} \hat{\mathcal{S}}_{10,11,12}\left[\frac{\delta_{\vec{k}'\vec{k}}}{\mathbf{A}_{r;\vec{0}}^{\vec{k}}}\right]\right]\right]\right]$$

$$\hat{\mathcal{S}}_{\dots} \equiv \hat{\mathcal{S}}_{\dots}^{\vec{\omega}^H,\, \mathcal{M}_\Delta^H}$$

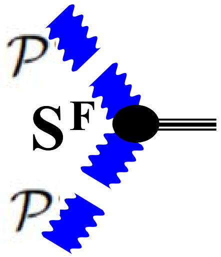
$$\mathbf{A}_i \equiv \mathbf{A}_{i;\,\vec{n}}^{\vec{k}'}$$

$$\{a_1,a_2\}_{rs} \equiv \begin{cases} \{10,11\} & r=\bar{s}' \\ \{10,12\} & r=\bar{s} \\ \{11,12\} & r=s' \end{cases} \qquad a_{rsj} \equiv \begin{cases} 10 & r=\bar{j},\, s=\bar{j}' \\ 11 & r=j',\, s=\bar{j}' \\ 12 & r=j',\, s=\bar{j} \end{cases}$$

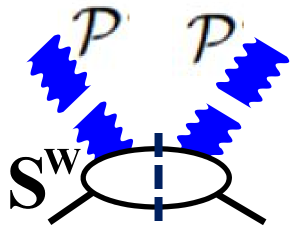
$$\bar{1}=2,\ \bar{2}=1 \qquad (1')'=1,\ (2')'=2$$

$$h_{\vec{n}}^{\vec{k}'} \equiv h_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})} = \frac{2^{\sum\limits_{\substack{r,s\in\Omega_H\\ r\neq s}}(k_{rs}-k'_{rs})-\sum\limits_{r\in\Omega_H} n_r}}{\prod\limits_{r\in\Omega_H}\left(c^{J_r}\right)_{n_r}}\mathbf{h}_{\vec{n}}^{\vec{k}'}$$

Other method to find solutions

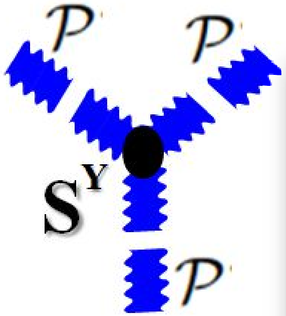


$$\bar{\mathcal{F}}^{J_1,J_2}_{k\ (\mu)_{J_1}(\nu)_{J_2}} = S^{F\ \{J_1,J_2\}}_{k,00\ (\alpha)_{J_1}(\beta)_{J_2}} \mathcal{P}^{J_1}_{(\alpha)_{J_1}(\mu)_{J_1}} \mathcal{P}^{J_2}_{(\beta)_{J_2}(\nu)_{J_2}}$$

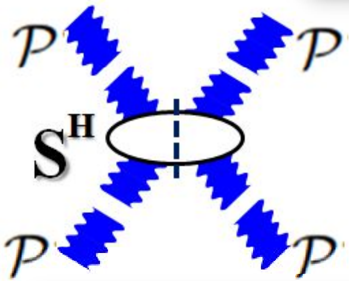


$$\bar{\mathcal{W}}^{J_1,J_{1'}}_{k\ (\mu)_{J_1}(\nu)_{J_2}} = S^{W\ \{J_1,J_{1'}\}}_{k,00\ (\alpha)_{J_1}(\beta)_{J_{1'}}} \mathcal{P}^{J_1}_{(\alpha)_{J_1}(\mu)_{J_1}} \mathcal{P}^{J_{1'}}_{(\beta)_{J_{1'}}(\mu')_{J_{1'}}}$$

Other method to find solutions



$$\bar{\mathcal{Y}}^{\vec{J}}_{\vec{k}\ (\mu)_{J_1}(\nu)_{J_2}(\rho)_{J_3}} = S^{Y;\ \vec{J}}_{\vec{k},\vec{0}\ (\alpha)_{J_1}(\alpha')_{J_2}(\alpha'')_{J_3}} \times$$
$$\times \mathcal{P}^{J_1}_{(\alpha)_{J_1}(\mu)_{J_1}} \mathcal{P}^{J_2}_{(\alpha')_{J_2}(\nu)_{J_2}} \mathcal{P}^{J_3}_{(\alpha'')_{J_3}(\rho)_{J_3}},$$



$$\bar{\mathcal{H}}^{\vec{J}}_{\vec{k}\ (\mu)_{J_1}(\mu')_{J_{1'}}(\nu)_{J_2}(\nu')_{J_{2'}}} = S^{H;\ \vec{J}}_{\vec{k},\vec{0}\ (\alpha)_{J_1}(\alpha')_{J_{1'}}(\beta)_{J_2}(\beta')_{J_{2'}}} \times$$
$$\times \mathcal{P}^{J_1}_{(\alpha)_{J_1}(\mu)_{J_1}} \mathcal{P}^{J_{1'}}_{(\alpha')_{J_{1'}}(\mu')_{J_{1'}}} \mathcal{P}^{J_2}_{(\beta)_{J_2}(\nu)_{J_2}} \mathcal{P}^{J_3}_{(\beta')_{J_{2'}}(\nu')_{J_{2'}}},$$

Non-conserved vertexes expansions

$$V^J \equiv \sum_{r=0}^{J} \overline{v}_r \left[\mathcal{V}^{J-r} q^r\right]$$

Symmetric Traceless

$$\left[\mathcal{V}^{J-r} q^r\right] = \sum_{u=0}^{[r/2]} \tilde{v}_u^J \left(\mathcal{V}_{(1)}^{J-r} q_{(1)}^r g_{(11)}^u\right)$$

Recurrence + solution

$$\tilde{v}_{u-1}^J - 2\tilde{v}_u^J(\tilde{c}^J + u - 1) = 0,$$

$$\tilde{c}^J = -(J + (D-4)/2).$$

$$\tilde{v}_u^J = \frac{q^{2u}}{2^u \left(\tilde{c}^J\right)_u}.$$

Orthogonality

$$\left[\mathcal{V}^{J-r} q^r\right] \otimes \left[\mathcal{V}^{J-s} q^s\right] = 0,\ r \neq s$$

Non-conserved vertexes expansions

$$W^{J_1,J_{1'}} = \sum_{r=0}^{J_1} \sum_{r'=0}^{J_{1'}} \overline{\omega}_{rr'} \left[\mathcal{W}_{(1),(1')}^{J_1-r,J_{1'}-r'} q_{(1)}^{r} q_{(1')}^{r'} \right]$$

Symmetric Traceless

$$\left[\mathcal{W}_{(1),(1')}^{J_1-r,J_{1'}-r'} q_{(1)}^{r} q_{(1')}^{r'} \right] = \sum_{u=0}^{[r/2]} \sum_{u'=0}^{[r'/2]} \tilde{\omega}_{u,u'}^{J_1,J_{1'}} \times \left(q_{(1)}^{r-2u} g_{(11)}^{u} \mathcal{W}_{(1),(1')}^{J_1-r,J_{1'}-r'} g_{(1'1')}^{u'} q_{(1')}^{r'-2u'} \right)$$

Orthogonality

$$\left[\mathcal{W}_{(1),(1')}^{J_1-r,J_{1'}-r'} q_{(1)}^{r} q_{(1')}^{r'} \right] \otimes \left[\mathcal{W}_{(1),(1')}^{J_1-s,J_{1'}-s'} q_{(1)}^{s} q_{(1')}^{s'} \right] = 0$$

$$r \neq s, r' \neq s';$$

Solution

$$\tilde{\omega}_{u,u'}^{J_1,J_{1'}} = \tilde{v}_{u}^{J_1} \tilde{v}_{u'}^{J_{1'}}$$

Non-conserved vertexes expansions

$$F^{J_1,J_2} \equiv \sum_{r=0}^{J_1}\sum_{s=0}^{J_2} \bar{f}_{rs}\left[\mathcal{F}^{J_1-r,J_2-s}_{(1),(2)} q^r_{1(1)} q^s_{2(2)}\right]$$

Symmetric Traceless

$$\left[\mathcal{F}^{J_1-r,J_2-s}_{(1),(2)} q^r_{1(1)} q^s_{2(2)}\right] = \sum_{u_1=0}^{[r/2]}\sum_{u_2=0}^{[s/2]} \tilde{f}^{J_1,J_2}_{u_1,u_2} \times \left(q^{r-2u_1}_{1(1)} g^{u_1}_{(11)} \mathcal{F}^{J_1-r,J_2-s}_{(1),(2)} g^{u_2}_{(22)} q^{s-2u_2}_{2(2)}\right)$$

Orthogonality

$$\left[\mathcal{F}^{J_1-r_1,J_2-r_2}_{(1),(2)} q^{r_1}_{1(1)} q^{r_2}_{2(2)}\right] \otimes \left[\mathcal{F}^{J_1-s_1,J_2-s_2}_{(1),(2)} q^{s_1}_{1(1)} q^{s_2}_{2(2)}\right] = 0$$

$$r_1 \neq s_1, r_2 \neq s_2$$

Solution

$$\tilde{f}^{J_1,J_2}_{u_1,u_2} = \tilde{v}^{J_1}_{u_1} \tilde{v}^{J_2}_{u_2}$$

Non-conserved vertexes expansions

$$Y^{\vec{J}} = \sum_{\vec{r}=\vec{0}}^{\vec{J}} \bar{y}_{\vec{r}} \left[\mathcal{Y}^{\vec{J}-\vec{r}}_{(1),(2),(3)} q^{r_1}_{1(1)} q^{r_2}_{2(2)} q^{r_3}_{3(3)} \right]$$

Orthogonality

$$\left[\mathcal{Y}^{\vec{J}-\vec{r}}_{(1),(2),(3)} q^{r_1}_{1(1)} q^{r_2}_{2(2)} q^{r_3}_{3(3)} \right] \otimes \left[\mathcal{Y}^{\vec{J}-\vec{s}}_{(1),(2),(3)} q^{s_1}_{1(1)} q^{s_2}_{2(2)} q^{s_3}_{3(3)} \right] = 0, \quad \vec{r} \neq \vec{s},$$

Symmetric Traceless

$$\left[\mathcal{Y}^{\vec{J}-\vec{r}}_{(1),(2),(3)} q^{r_1}_{1(1)} q^{r_2}_{2(2)} q^{r_3}_{3(3)} \right] = \sum_{\vec{u}=\vec{0}}^{[\vec{r}/2]} \tilde{y}^{\vec{J}}_{\vec{u}} \left(\mathcal{Y}^{\vec{J}-\vec{r}}_{(1),(2),(3)} \, g^{u_1}_{(11)} q^{r_1-2u_1}_{1(1)} \times g^{u_2}_{(22)} q^{i_2-2u_2}_{2(2)} g^{u_3}_{(33)} q^{i_3-2u_3}_{3(3)} \right)$$

Solution

$$\tilde{y}^{\vec{J}}_{\vec{u}} = \prod_{r \in \Omega_Y} \tilde{v}^{J_r}_{u_r}$$

Non-conserved vertexes expansions

$$H^{\vec{J}} \equiv \sum_{\vec{r}=\vec{0}}^{\vec{J}} \bar{h}_{\vec{r}} \left[\mathcal{H}^{\vec{J}-\vec{r}}_{(1),(1'),(2),(2')} q^{r_1}_{1(1)} q^{r_{1'}}_{1(1')} q^{r_2}_{2(2)} q^{r_{2'}}_{2(2')} \right]$$

$$\left[\mathcal{H}^{\vec{J}-\vec{r}}_{(1),(1'),(2),(2')} q^{r_1}_{1(1)} q^{r_{1'}}_{1(1')} q^{r_2}_{2(2)} q^{r_{2'}}_{2(2')} \right] = \sum_{\vec{u}=\vec{0}}^{[\vec{r}/2]} \tilde{h}^{\vec{J}}_{\vec{u}} \left(\mathcal{H}^{\vec{J}-\vec{r}}_{(1),(1'),(2),(2')} \, g^{u_1}_{(11)} q^{r_1-2u_1}_{1(1)} \times g^{u_{1'}}_{(1'1')} q^{i_{1'}-2u_{1'}}_{1(1')} g^{u_2}_{(22)} q^{i_2-2u_2}_{2(2)} g^{u_{2'}}_{(2'2')} q^{i_{2'}-2u_{2'}}_{2(2')} \right)$$

Symmetric Traceless

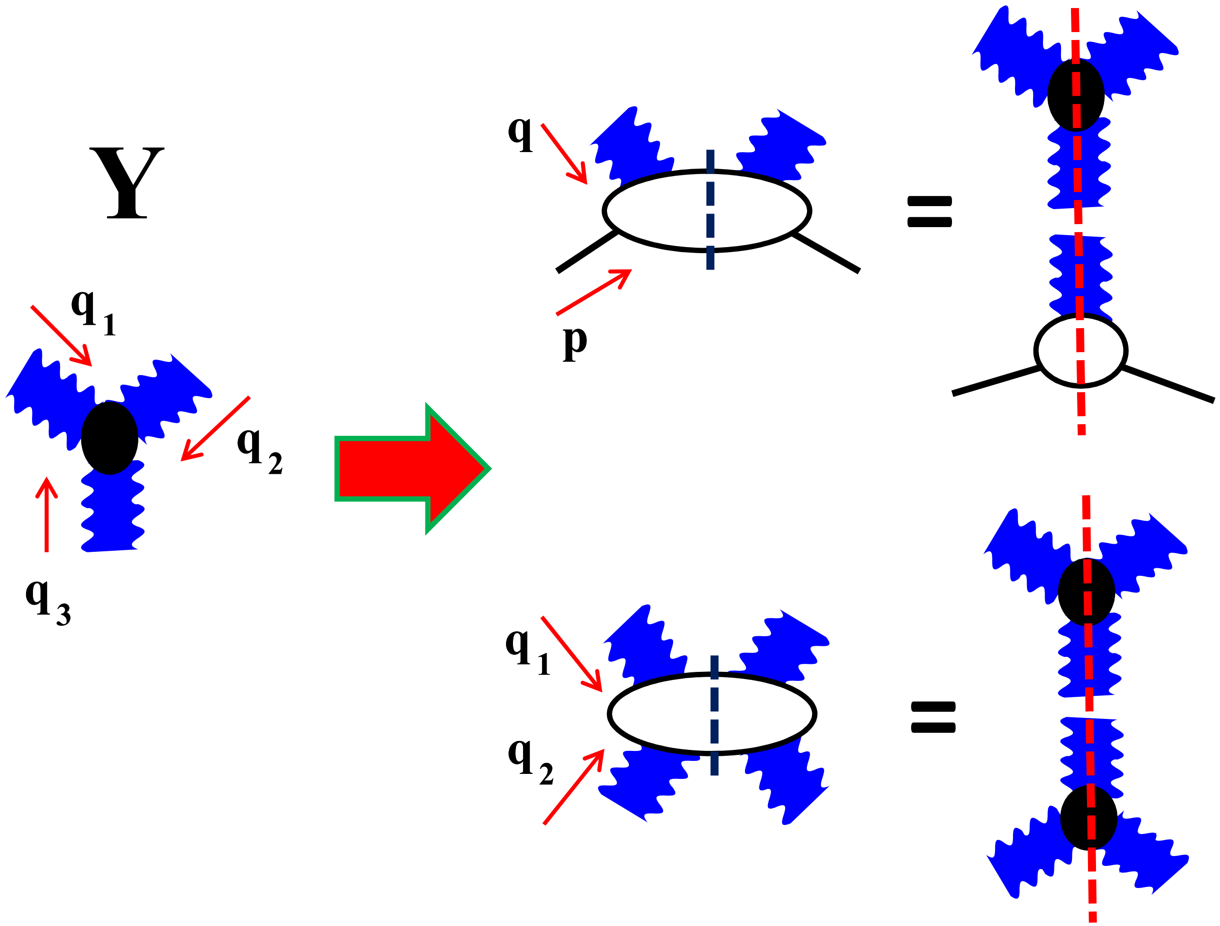
Solution

$$\tilde{h}^{\vec{J}}_{\vec{u}} = \prod_{r \in \Omega_H} \tilde{v}^{J_r}_{u_r}$$

Orthogonality

$$\left[\mathcal{H}^{\vec{J}-\vec{r}}_{(1),(1'),(2),(2')} q^{r_1}_{1(1)} q^{r_{1'}}_{1(1')} q^{r_2}_{2(2)} q^{r_{2'}}_{2(2')} \right] \otimes \left[\mathcal{H}^{\vec{J}-\vec{s}}_{(1),(1'),(2),(2')} q^{s_1}_{1(1)} q^{s_{1'}}_{1(1')} q^{s_2}_{2(2)} q^{s_{2'}}_{2(2')} \right] = 0, \; \vec{r} \neq \vec{s}.$$

3-Reggeon vertex contractions



Further study

- Spin-parity analysis in CEDP
- Half-integer spins, helicity amplitudes
- Curved spaces, "non-conserved" currents
- Hyper-fields, strings
- Non-diffractive processes
- Unitarization
- Wolfram Mathematica package
- Universal framework for diffraction

To be continued …

THANK YOU !

BACKUP SLIDES

Transverse Symmetric tensors (Y)

$$\mathcal{K} \equiv (q_i q_j)^2 - q_i^2 q_j^2 = \frac{1}{4}\lambda(q_1^2, q_2^2, q_3^2)$$

$$Q_i^2 = -q_i^2 \equiv -t_i,\ i,j \in \{1,2,3\}, ; i \neq j,$$

$$q_3 + q_1 + q_2 = 0,$$

$$P_{i\ \alpha} \equiv \frac{(q_i q_s) q_{r\ \alpha} - (q_i q_r) q_{s\ \alpha}}{Q_i \sqrt{|\mathcal{K}|}},$$

$$P_{i\ \alpha} q_{i\ \alpha} = 0.$$

$$\chi_{ij} = \frac{Q_i Q_j}{|q_i q_j|},\ \lambda_{ij} = 1 - \kappa_i \kappa_j \chi_{ij}^2, \qquad \tilde{\lambda}_{ir}^r \equiv \kappa_r |\lambda_{ir}|,$$

$$\hat{\lambda}_{rs}^i \equiv \prod_{\substack{a \neq b \\ a,b \in \{1,2,3\}}} \kappa_{ab} \cdot \eta_{is} \eta_{ir} \chi_{rs} \sqrt{|\lambda_{ir}||\lambda_{is}|}, \qquad \bar{\chi}_{ij} = \kappa_i \kappa_{ij} \chi_{ij} \bar{\eta}_{ij}.$$

$$\kappa_{ij} = \mathrm{sign}[q_i q_j],\ \kappa_i = \mathrm{sign}[-q_i^2],\ \kappa = \mathrm{sign}[\mathcal{K}],$$

$$\eta_{ij} \equiv \begin{cases} +1 & i > j \\ -1 & i < j \end{cases},\ \bar{\eta}_{ij} \equiv \begin{cases} \eta_{si} & j = r \\ \eta_{ir} & j = s \\ r < s, & r, s \neq i \end{cases},$$

$$r, s \in \{1,2,3\},$$

$$Q_1 P_1 + Q_3 P_3 = Q_2 P_2,$$

$$Q_s \frac{\sqrt{|\lambda_i s|}}{\chi_{is}} = Q_r \frac{\sqrt{|\lambda_i r|}}{\chi_{ir}},$$

$$P_i^2 = \kappa \kappa_i \equiv \bar{\kappa}_i.$$

Irreducible tensors (spur S^V)

$$Sp_1 S_{n_1}^{V;J_1} = g_{\mu_1\mu_2}\left[P_{(1)}^{\mu_1}P_{(1)}^{\mu_2}S_{n_1}^{V;J_1-2}+ \right.$$
$$+\sum_i \left(P_{(1)}^{\mu_1}G_{(11)}^{\mu_2\mu_i}+P_{(1)}^{\mu_2}G_{(11)}^{\mu_1\mu_i}\right)S_{n_1-1}^{V;J_1-3}+$$
$$\left.+\sum_{i<j}G_{(11)}^{\mu_1\mu_i}G_{(11)}^{\mu_2\mu_j}S_{n_1-2}^{V;J_1-4}+G_{(11)}^{\mu_1\mu_2}S_{n_1-1}^{V;J_1-2}\right]$$

$$\mathcal{N}_n^J = \frac{J!}{2^n n!(J-2n)!}$$

$$S_{n_1}^{V;J_1-2} + 2\frac{(J_1-2)\mathcal{N}_{n_1-1}^{J_1-3}}{\mathcal{N}_{n_1-1}^{J_1-2}}S_{n_1-1}^{V;J_1-2} +$$
$$+\frac{(J_1-2)(J_1-3)\mathcal{N}_{n_1-2}^{J_1-4}}{\mathcal{N}_{n_1-1}^{J_1-2}}S_{n_1-1}^{V;J_1-2} + (D-1)S_{n_1-1}^{V;J_1-2} =$$
$$S_{n_1}^{V;J_1-2} + (2(J_1-2n_1)+2(n_1-1)+D-1)S_{n_1-1}^{V;J_1-2},$$

Generating function method (Y)

$$\bar{\mathcal{Y}}_g(\{x_r\},\{y_r\},\{y_{rs}\}) = \bar{\mathcal{Y}}^{\vec{J}} \prod_{r\in\Omega_Y} \omega_{(r)}^{J_r} =$$

$$= \sum_{\tilde{\Omega}^Y_{\vec{k}',\vec{n}}} y_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})} \mathcal{N}^{Y\ \vec{J}}_{\vec{k},\vec{n}} \times \prod_{r\in\Omega_Y} x_r^{\bar{J}_r} y_r^{n_r} \prod_{\substack{\bar{r}\neq\bar{s} \\ \bar{r},\bar{s}\in\Omega_Y}} y_{\bar{r}\bar{s}}^{k'_{\bar{r}\bar{s}}}$$

$$x_r = P_r\omega_r,\ y_r = \omega_r^2 - \frac{(\omega_r q_r)^2}{q_r^2},$$

$$y_{rs} = \omega_r\omega_s - \frac{(\omega_s q_r)(\omega_r q_s)}{q_r q_s},\ y_{rs} \equiv y_{sr},\ r \neq s$$

$$\mathcal{N}^{Y\ \vec{J}}_{\vec{k}',\vec{n}} = \prod_{r\in\Omega_Y} \frac{J_r!}{2^{n_r} n_r! \bar{J}_r} \prod_{\substack{\bar{r}\neq\bar{s} \\ \bar{r},\bar{s}\in\Omega_Y}} \frac{1}{k'_{\bar{r}\bar{s}}!}$$

Generating function method (Y)

$$g_{\alpha\beta}\partial_{\omega_{i\alpha}}\partial_{\omega_{i\beta}} =$$
$$\bar{\kappa}_i\partial^2_{x_i x_i} + 4x_i\partial^2_{x_i y_i} + 4y_i\partial^2_{y_i y_i} + 2\sum_{r\neq i}\bar{\chi}_{ir}x_r\partial^2_{x_i y_{ir}} +$$
$$+4\sum_{r\neq i} y_{ir}\partial^2_{y_i y_{ir}} + \sum_{r\neq i}(y_r - \tilde{\lambda}^r_{ir}x_r^2)\partial^2_{y_{ir}y_{ir}} +$$
$$+2(y_{rs} + \hat{\lambda}^i_{rs}x_r x_s)_{r\neq s\neq i}\partial^2_{y_{ir}y_{is}} + 2(D-1)\partial_{y_i}.$$

Generating function method (Y)

$$y_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})}\mathcal{N}^{Y\ \vec{J}}_{\vec{k}',\vec{n}}\Big[\bar{\kappa}_i\bar{J}_i\left(\bar{J}_i-1\right)x_i^{\bar{J}_i-2}...+4\bar{J}_i n_i y_i^{n_i-1}...+$$

$$+4n_i(n_i-1)y_i^{n_i-1}...+2\bar{J}_i\sum_{r\neq i}\bar{\chi}_{ir}k'_{ir}x_i^{\bar{J}_i-1}x_r^{\bar{J}_r+1}y_{ir}^{k'_{ir}-1}...+$$

$$+4n_i\sum_{r\neq i}k'_{ir}y_i^{n_i-1}...+$$

$$+\sum_{r\neq i}k'_{ir}(k'_{ir}-1)\left(y_r^{n_r+1}-\tilde{\lambda}^r_{ir}x_r^{\bar{J}_r+2}\right)y_{ir}^{k'_{ir}-2}...+$$

$$+2k'_{ir}k'_{is}\left(y_{rs}^{k'_{rs}+1}+\hat{\lambda}^i_{rs}x_r^{\bar{J}_r+1}x_s^{\bar{J}_s+1}\right)_{r\neq s\neq i}y_{ir}^{k'_{ir}-1}y_{is}^{k'_{is}-1}...+$$

$$+2(D-1)n_i y_i^{n_i-1}...\Big]=0,$$

Generating function method (H)

$$\bar{\mathcal{H}}_g(\{x_r\},\{y_r\},\{y_{rs}\}) = \bar{\mathcal{H}}^{\vec{J}} \prod_{r\in\Omega_H} \omega_{(r)}^{J_r} =$$

$$\sum_{\tilde{\Omega}^H_{\vec{k}',\vec{n}}} h_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})} \mathcal{N}^{H\ \vec{J}}_{\vec{k},\vec{n}} \times \prod_{r\in\Omega_H} x_r^{\bar{J}_r} y_r^{n_r} \prod_{\substack{\hat{r}\neq\hat{s}\\ \hat{r},\hat{s}\in\Omega_H}} y_{\hat{r}\hat{s}}^{k'_{\hat{r}\hat{s}}}$$

$$x_r = P_r\omega_r,\ y_r = \omega_r^2 - \frac{(\omega_r q_r)^2}{q_r^2},$$

$$y_{rs} = \omega_r\omega_s - \frac{(\omega_s q_r)(\omega_r q_s)}{q_r q_s},\ y_{rs} \equiv y_{sr},\ r \neq s,$$

$$y_{ii'} = \omega_i\omega_{i'} - \frac{(\omega_i q_i)(\omega_{i'} q_i)}{q_i^2},\ i = 1,2.$$

$$\mathcal{N}^{H\ \vec{J}}_{\vec{k}',\vec{n}} = \prod_{r\in\Omega_H} \frac{J_r!}{2^{n_r} n_r! \bar{J}_r} \prod_{\substack{\bar{r}\neq\bar{s}\\ \bar{r},\bar{s}\in\Omega_H}} \frac{1}{k'_{\bar{r}\bar{s}}!},$$

Generating function method (H)

$$g_{\alpha\beta}\partial_{\omega_{(i)\alpha}}\partial_{\omega_{(i)\beta}} =$$

$$\partial^2_{x_i x_i} + 4x_i\partial^2_{x_i y_i} + 4y_i\partial^2_{y_i y_i} + 2x_{i'}\partial^2_{x_i y_{ii'}} +$$

$$+2\chi_m \sum_{r\neq i,i'} x_r\partial^2_{x_i y_{ir}} + 4\sum_{r\neq i} y_{ir}\partial^2_{y_i y_{ir}} + y_{i'}\partial^2_{y_{ii'} y_{ii'}} +$$

$$+\sum_{r\neq i,i'} (y_r - \lambda_m x_r^2)\partial^2_{y_{ir} y_{ir}} + 2(y_{jj'} - \lambda_m x_j x_{j'})\partial^2_{y_{ij} y_{ij'}} +$$

$$+\sum_{r\neq i,i'} 2y_{i'r}\partial^2_{y_{ii'} y_{ir}} + 2(D-1)\partial_{y_i},$$

$$\{j, j'\} \neq \{i, i'\},\ j \equiv \bar{i},\ j' \equiv \bar{i}'.$$

Generating function method (H)

$$h_{\vec{n}}^{\vec{k}',(\vec{k};\vec{J})}\mathcal{N}_{\vec{k},\vec{n}}^{H\ \vec{J}}\Big[\bar{J}_i\left(\bar{J}_i-1\right)x_i^{\bar{J}_i-2}...+4\bar{J}_i n_i y_i^{n_i-1}...+$$

$$+4n_i(n_i-1)y_i^{n_i-1}...+2\bar{J}_i k'_{ii'}x_i^{\bar{J}_i-1}x_{i'}^{\bar{J}_{i'}+1}y_{ii'}^{k'_{ii'}-1}...+$$

$$+2\chi_m\bar{J}_i\sum_{r\neq i,i'}k'_{ir}x_i^{\bar{J}_i-1}x_r^{\bar{J}_r+1}y_{ir}^{k'_{ir}-1}...+$$

$$+4n_i\sum_{r\neq i}k'_{ir}y_i^{n_i-1}...+k'_{ii'}(k'_{ii'}-1)y_{i'}^{n_{i'}+1}y_{ii'}^{k'_{ii'}-2}...+$$

$$+\sum_{r\neq i,i'}k'_{ir}(k'_{ir}-1)\left(y_r^{n_r+1}-\lambda_m x_r^{\bar{J}_r+2}\right)y_{ir}^{k'_{ir}-2}...+$$

$$+2k'_{ij}k'_{ij'}\left(y_{jj'}^{k'_{jj'}+1}-\lambda_m x_j^{\bar{J}_j+1}x_{j'}^{\bar{J}_{j'}+1}\right)y_{ij}^{k'_{ij}-1}y_{ij'}^{k'_{ij'}-1}...+$$

$$+\sum_{r\neq i,i'}2k'_{ii'}k'_{ir}y_{ii'}^{k'_{ii'}-1}y_{ir}^{k'_{ir}-1}y_{i'r}^{k'_{i'r}+1}...+$$

$$+2(D-1)n_i y_i^{n_i-1}...\Big]=0,$$