Rephasing invariants of CP violation

for light and heavy Majorana neutrinos

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in general, the bigger a mountain the older it is

*Based on **ZZX**, 2505.02415 (PLB); **S. Luo**, **ZZX**, 2508.15662*

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OUTLINE

- All the roads lead to Rome: quartets and trios
- Invariants of leptonic CP violation on seesaw
- Heavy flavor decays / light flavor oscillations

On the basis of weak interactions

★ **In the standard electroweak model the weak-interaction flavor basis involves some arbitrariness.**

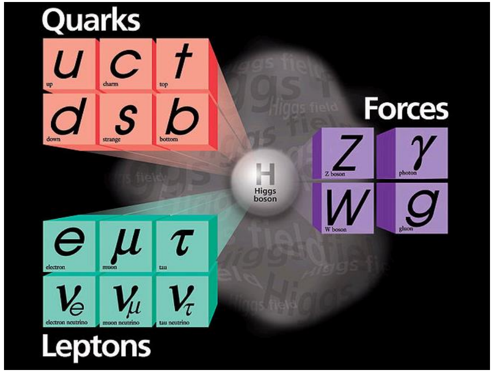
$$\mathcal{L}_{\rm SM} = \mathcal{L}_{\rm gauge} + \mathcal{L}_{\rm Higgs} + \mathcal{L}_{\rm fermion} + \mathcal{L}_{\rm Yukawa}$$

$$\mathcal{L}_{\rm G} = -\frac{1}{4}\left(W^{i\mu\nu}W^{i}_{\mu\nu} + B^{\mu\nu}B_{\mu\nu}\right)$$

$$\mathcal{L}_{\rm H} = (D^{\mu}H)^{\dagger}(D_{\mu}H) - \mu^{2}H^{\dagger}H - \lambda(H^{\dagger}H)^{2}$$

$$\mathcal{L}_{\rm F} = \overline{Q_{\rm L}}{\rm i}\not{D}Q_{\rm L} + \overline{\ell_{\rm L}}{\rm i}\not{D}\ell_{\rm L} + \overline{U_{\rm R}}{\rm i}\not{\partial}' U_{\rm R} + \overline{D_{\rm R}}{\rm i}\not{\partial}' D_{\rm R} + \overline{E_{\rm R}}{\rm i}\not{\partial}' E_{\rm R}$$

$$\mathcal{L}_{\rm Y} = -\overline{Q_{\rm L}}Y_{\rm u}\widetilde{H}U_{\rm R} - \overline{Q_{\rm L}}Y_{\rm d}HD_{\rm R} - \overline{\ell_{\rm L}}Y_{l}HE_{\rm R} + {\rm h.c.}$$



One may choose to define some flavor-basis-independent invariants so as to describe flavor mixing and CP violation without introducing any specific parametrizations and phase conventions.

★ **There exists a unique basis of weak interactions —— the mass basis of all the quarks and leptons. In this basis one may simply consider the rephasing invariants of flavor mixing and CP violation.**

Why weak CP violation exists

★ **Reason 1: the quark fields interact, simultaneously but in different ways, with the Higgs and gauge fields.**

★ **Reason 2: the quarks have three different families.**

} **the Kobayashi-Maskawa mechanism**

★ **In the flavor basis, quark masses, flavor mixing and CP violation originate from the complex and non-diagonal Yukawa interactions.**

$$C_q \equiv \mathrm{i}\left[M_\mathrm{u} M_\mathrm{u}^\dagger , M_\mathrm{d} M_\mathrm{d}^\dagger\right] \longrightarrow \det C_q = -2\mathcal{J}_q \left(m_u^2 - m_c^2\right)\left(m_c^2 - m_t^2\right)\left(m_t^2 - m_u^2\right)\left(m_d^2 - m_s^2\right)\left(m_s^2 - m_b^2\right)\left(m_b^2 - m_d^2\right)$$

Jarlskog invariant: $\mathcal{J}_q = \frac{1}{8} \sin 2\vartheta_{12} \sin 2\vartheta_{13} \sin 2\vartheta_{23} \cos\vartheta_{13} \sin\delta_q$

Similar to QM?

- **W. Heisenberg, ZPC 33 (1925) 879**
- **M. Born, P. Jordan, ZPC 34 (1925) 858**
- **M. Born, W. Heisenberg, P. Jordan, ZPC 35 (1926) 557**

$$\left[\hat{x}(t), \hat{p}(t)\right] = \mathrm{i}\hbar$$

★ **In the mass basis, flavor mixing and CP violation are described by a 3 × 3 unitary matrix with an irremovable KM phase in the weak charged-current interactions.**

J-invariant of CP violation

★ In the mass basis of quarks, it is the 3 × 3 CKM matrix that characterizes flavor mixing and weak CP violation of the SM. The only but powerful constraint on this matrix is unitarity:

$$-\mathcal{L}_{\rm CC} = \frac{g}{\sqrt{2}} \overline{(u, \quad c, \quad t)_{\rm L}} \gamma^\mu \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{\rm L} W^+_\mu + {\rm h.c.}$$

N. Cabibbo (1963), M. Kobayashi, T. Maskawa (1973)

unitarity

$$\begin{cases} \sum_i V_{\alpha i} V^*_{\beta i} = \delta_{\alpha\beta} \\ \sum_\alpha V_{\alpha i} V^*_{\alpha j} = \delta_{ij} \end{cases}$$

★ The strength of CP violation in the SM is measured by a universal rephasing invariant:

$$\mathcal{J}_q \sum_\gamma \varepsilon_{\alpha\beta\gamma} \sum_k \varepsilon_{ijk} = {\rm Im}\left(V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}\right)$$

(C. Jarlskog 1985, D.d. Wu 1986)

The commutator and the J-invariant contain the same information on CP violation (H. Fritzsch, ZZX, 1999)

★ The CKM quartets are rephasing-invariant as they are insensitive to the phase transformations:

$$q_\alpha \to e^{{\rm i}\phi_\alpha} q_\alpha \; , \quad q_\beta \to e^{{\rm i}\phi_\beta} q_\beta \; ; \quad q_i \to e^{{\rm i}\phi_i} q_i \; , \quad q_j \to e^{{\rm i}\phi_j} q_j \; .$$

$$V_{\alpha i} \to e^{{\rm i}(\phi_\alpha - \phi_i)} V_{\alpha i} \; , \quad V_{\beta j} \to e^{{\rm i}(\phi_\beta - \phi_j)} V_{\beta j} \; , \quad V_{\alpha j} \to e^{{\rm i}(\phi_\alpha - \phi_j)} V_{\alpha j} \; , \quad V_{\beta i} \to e^{{\rm i}(\phi_\beta - \phi_i)} V_{\beta i} \; .$$

These phases cancelled out

New invariants built from the CKM trios

★ **New rephrasing invariants in terms of the CKM trios (ZZX, 2508.12589; S. Luo, ZZX, 2508.15662):**

$$\diamondsuit^{ijk}_{\alpha\beta\gamma} \equiv \frac{1}{\det V}\left(V_{\alpha i} V_{\beta j} V_{\gamma k}\right)$$

$$\alpha \neq \beta \neq \gamma \text{ and } i \neq j \neq k$$

$$\left\{\begin{array}{ll} q_\alpha \to q_\alpha e^{\mathrm{i}\phi_\alpha} \, , & q_i \to q_i e^{\mathrm{i}\phi_i} \\ q_\beta \to q_\beta e^{\mathrm{i}\phi_\beta} \, , & q_j \to q_j e^{\mathrm{i}\phi_j} \\ q_\gamma \to q_\gamma e^{\mathrm{i}\phi_\gamma} \, , & q_k \to q_k e^{\mathrm{i}\phi_k} \end{array}\right.$$

$$\longrightarrow \left\{\begin{array}{l} V_{\alpha i} \to V_{\alpha i} e^{\mathrm{i}\left(\phi_\alpha - \phi_i\right)} \, , \quad V_{\beta j} \to V_{\beta j} e^{\mathrm{i}\left(\phi_\beta - \phi_j\right)} \, , \quad V_{\gamma k} \to V_{\gamma k} e^{\mathrm{i}\left(\phi_\gamma - \phi_k\right)} \\ \det V \to e^{\mathrm{i}\left(\phi_\alpha + \phi_\beta + \phi_\gamma\right)} \left(\det V\right) e^{-\mathrm{i}\left(\phi_i + \phi_j + \phi_k\right)} \end{array}\right.$$

$$\begin{aligned} \diamondsuit^{ijk}_{\alpha\beta\gamma} = V_{\alpha i} V_{\beta j} V_{\gamma k} \left(\det V\right)^* &= \left[|V_{\alpha i}|^2 |V_{\beta j}|^2 - V_{\alpha i} V_{\beta j} V^*_{\alpha j} V^*_{\beta i}\right] \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} \\ &= \left[|V_{\alpha i}|^2 |V_{\gamma k}|^2 - V_{\alpha i} V_{\gamma k} V^*_{\alpha k} V^*_{\gamma i}\right] \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} \\ &= \left[|V_{\beta j}|^2 |V_{\gamma k}|^2 - V_{\beta j} V_{\gamma k} V^*_{\beta k} V^*_{\gamma j}\right] \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} \end{aligned}$$

★ **Then we arrive in Rome in a trio way:**

$$\mathrm{Im}\,\diamondsuit^{ijk}_{\alpha\beta\gamma} = -\mathcal{J}_q$$

New invariants built from the PMNS trios

★ **New leptonic rephrasing invariants in terms of the PMNS trios (S. Luo, ZZX, 2508.15662):**

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(e \ \mu \ \tau)_{\rm L}} \gamma^\mu U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} W^-_\mu + {\rm h.c.}$$

$$\blacklozenge^{ijk}_{\alpha\beta\gamma} \equiv \frac{1}{\det U} \left(U_{\alpha i} U_{\beta j} U_{\gamma k} \right) , \qquad \alpha \neq \beta \neq \gamma \text{ and } i \neq j \neq k$$

★ **But the origin of tiny neutrino masses has be taken into account, as *U* is likely to be non-unitary.**

OUTLINE

- ♦ All the roads lead to Rome: quartets and trios
- ♦ Invariants of leptonic CP violation on seesaw
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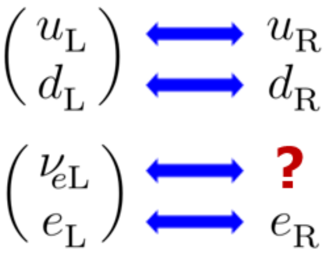
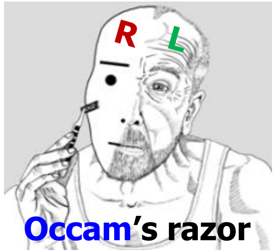
Seesaw: a most natural extension of the SM

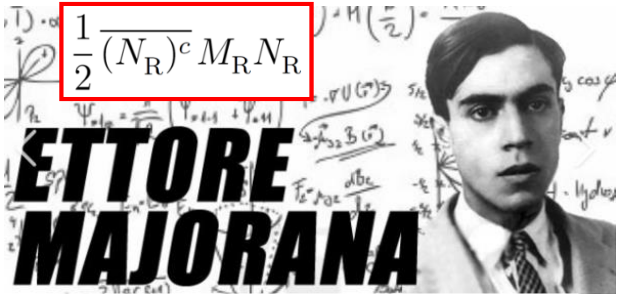
- ♦ Neutrinos surely have the *right* to be *right* (-handed) to keep a similar kind of left-right symmetry as charged leptons and quarks —— small animals' fair play?





- ♦ Then neutrinos are allowed to couple to the SM Higgs doublet —— the Yukawa interactions. Why not?

- ♦ But the gender of neutrinos (neutral) makes it very fair to add a Majorana mass term with N and N^c, which is fully *harmless* to all the fundamental symmetries of the SM.



- ♦ Then we are led to seesaw (P. Minkowski 1977), a mechanism consistent with Weinberg's SMEFT (1979).

$$-\mathcal{L}_{\rm lepton} = \overline{\ell_L}Y_l H l_R + \overline{\ell_L}Y_\nu \widetilde{H} N_R + \frac{1}{2}\overline{(N_R)^c}M_R N_R + {\rm h.c.}$$

$$= \overline{l_L}Y_l l_R \phi^0 + \frac{1}{2}\overline{\left[\nu_L \quad (N_R)^c\right]}\begin{pmatrix} 0 & Y_\nu \phi^{0*} \\ Y_\nu^T \phi^{0*} & M_R \end{pmatrix}\begin{bmatrix} (\nu_L)^c \\ N_R \end{bmatrix} + \overline{\nu_L}Y_l l_R \phi^+ - \overline{l_L}Y_\nu N_R \phi^- + {\rm h.c.}$$

CP violation in heavy neutrino decays is crucial for leptogenesis (M. Fukugita, T. Yanagiada 1986).

Two types of lepton flavor mixing

♦ A basis transformation to obtain **Majorana** neutrino masses and flavor mixing before or after **SSB**.

$$\mathbb{U}^\dagger \begin{pmatrix} \mathbf{0} & Y_\nu \phi^{0*} \\ Y_\nu^T \phi^{0*} & M_{\rm R} \end{pmatrix} \mathbb{U}^* = \begin{pmatrix} D_\nu & \mathbf{0} \\ \mathbf{0} & D_N \end{pmatrix}$$

working masses:
$$\begin{cases} D_\nu \equiv {\rm Diag}\{m_1, m_2, m_3\} \\ D_N \equiv {\rm Diag}\{M_4, M_5, M_6\} \end{cases}$$

Decomposition: $\mathbb{U} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & U_0' \end{pmatrix} \begin{pmatrix} A & R \\ S & B \end{pmatrix} \begin{pmatrix} U_0 & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$

sterile (unitary)	Yukawa (interplay)	active (unitary)

A block parameterization (ZZX, 1110.0083)

♦ **Weak charged-current interactions of leptons in the seesaw mechanism:**

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(e \quad \mu \quad \tau)_{\rm L}} \gamma^\mu \left[U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} + R \begin{pmatrix} N_4 \\ N_5 \\ N_6 \end{pmatrix}_{\rm L} \right] W_\mu^- + {\rm h.c.}$$

oscillations, LNV ⟵ light (U) heavy (R) ⟶ collider, LNV, LFV

$U = AU_0$: the **PMNS** matrix

R : an analogue for heavy

$$U D_\nu U^T = ({\rm i}R)\, D_N \,({\rm i}R)^T$$

⟵ $UU^\dagger + RR^\dagger = I$ (unitarity relation)

- The **PMNS** matrix U is not exactly **unitary** in the seesaw scenario
- But **non-unitarity** of U is constrained to be very small

The full Euler-like parametrization

♦ The **1st** full **Euler-like** parametrization of $U = AU_0$ and R is useful for calculating flavor structures.

$$U_0 = \begin{pmatrix} c_{12}c_{13} & \hat{s}^*_{12}c_{13} & \hat{s}^*_{13} \\ -\hat{s}_{12}c_{23} - c_{12}\hat{s}_{13}\hat{s}^*_{23} & c_{12}c_{23} - \hat{s}^*_{12}\hat{s}_{13}\hat{s}^*_{23} & c_{13}\hat{s}^*_{23} \\ \hat{s}_{12}\hat{s}_{23} - c_{12}\hat{s}_{13}c_{23} & -c_{12}\hat{s}_{23} - \hat{s}^*_{12}\hat{s}_{13}c_{23} & c_{13}c_{23} \end{pmatrix}$$

derivable from the parameters of A and R

$$A = \begin{pmatrix} c_{14}c_{15}c_{16} & 0 & 0 \\ -c_{14}c_{15}\hat{s}_{16}\hat{s}^*_{26} - c_{14}\hat{s}_{15}\hat{s}^*_{25}c_{26} - \hat{s}_{14}\hat{s}^*_{24}c_{25}c_{26} & c_{24}c_{25}c_{26} & 0 \\ -c_{14}c_{15}\hat{s}_{16}c_{26}\hat{s}^*_{36} + c_{14}\hat{s}_{15}\hat{s}^*_{25}\hat{s}_{26}\hat{s}^*_{36} - c_{14}\hat{s}_{15}c_{25}\hat{s}^*_{35}c_{36} + \hat{s}_{14}\hat{s}^*_{24}c_{25}\hat{s}_{26}\hat{s}^*_{36} + \hat{s}_{14}\hat{s}^*_{24}\hat{s}_{25}\hat{s}^*_{35}c_{36} - \hat{s}_{14}c_{24}\hat{s}^*_{34}c_{35}c_{36} & -c_{24}c_{25}\hat{s}_{26}\hat{s}^*_{36} - c_{24}\hat{s}_{25}\hat{s}^*_{35}c_{36} - \hat{s}_{24}\hat{s}^*_{34}c_{35}c_{36} & c_{34}c_{35}c_{36} \end{pmatrix}$$

$$R = \begin{pmatrix} \hat{s}^*_{14}c_{15}c_{16} & \hat{s}^*_{15}c_{16} & \hat{s}^*_{16} \\ -\hat{s}^*_{14}c_{15}\hat{s}_{16}\hat{s}^*_{26} - \hat{s}^*_{14}\hat{s}_{15}\hat{s}^*_{25}c_{26} + c_{14}\hat{s}^*_{24}c_{25}c_{26} & -\hat{s}^*_{15}\hat{s}_{16}\hat{s}^*_{26} + c_{15}\hat{s}^*_{25}c_{26} & c_{16}\hat{s}^*_{26} \\ -\hat{s}^*_{14}c_{15}\hat{s}_{16}c_{26}\hat{s}^*_{36} + \hat{s}^*_{14}\hat{s}_{15}\hat{s}^*_{25}\hat{s}_{26}\hat{s}^*_{36} - \hat{s}^*_{14}\hat{s}_{15}c_{25}\hat{s}^*_{35}c_{36} - c_{14}\hat{s}^*_{24}c_{25}\hat{s}_{26}\hat{s}^*_{36} - c_{14}\hat{s}^*_{24}\hat{s}_{25}\hat{s}^*_{35}c_{36} + c_{14}c_{24}\hat{s}^*_{34}c_{35}c_{36} & -\hat{s}^*_{15}\hat{s}_{16}c_{26}\hat{s}^*_{36} - c_{15}\hat{s}^*_{25}\hat{s}_{26}\hat{s}^*_{36} + c_{15}c_{25}\hat{s}^*_{35}c_{36} & c_{16}c_{26}\hat{s}^*_{36} \end{pmatrix}$$

ZZX
0709.2220/1110.0083

The latest stringent *bounds* on possible PMNS nonunitarity.
M. Blennow et al. 2023

$$\begin{cases} \theta_{1j} < 2.92^\circ \\ \theta_{2j} < 0.27^\circ \\ \theta_{3j} < 2.56^\circ \\ [j = 4, 5, 6] \end{cases}$$

ZZX, J. Zhu, 2412.17698

Non-unitarity of the PMNS matrix

♦ The **PMNS** matrix $U = AU_0$ in the seesaw mechanism is *non-unitary*, but this effect is rather small.

$$A = I - \frac{1}{2}\begin{pmatrix} s_{14}^2 + s_{15}^2 + s_{16}^2 & 0 & 0 \\ 2\hat{s}_{14}\hat{s}_{24}^* + 2\hat{s}_{15}\hat{s}_{25}^* + 2\hat{s}_{16}\hat{s}_{26}^* & s_{24}^2 + s_{25}^2 + s_{26}^2 & 0 \\ 2\hat{s}_{14}\hat{s}_{34}^* + 2\hat{s}_{15}\hat{s}_{35}^* + 2\hat{s}_{16}\hat{s}_{36}^* & 2\hat{s}_{24}\hat{s}_{34}^* + 2\hat{s}_{25}\hat{s}_{35}^* + 2\hat{s}_{26}\hat{s}_{36}^* & s_{34}^2 + s_{35}^2 + s_{36}^2 \end{pmatrix} + \mathcal{O}\left(s_{ij}^4\right)$$

So $U = AU_0 = U_0 + \text{nonunitarity corrections } (\lesssim 10^{-3})$.

♦ The strength of **Yukawa interactions** is proportional to ***R***.
And **CP violation** in heavy **Majorana** neutrino decays and **thermal leptogenesis** are determined by nonzero ***R***.

$$R = \begin{pmatrix} \hat{s}_{14}^* & \hat{s}_{15}^* & \hat{s}_{16}^* \\ \hat{s}_{24}^* & \hat{s}_{25}^* & \hat{s}_{26}^* \\ \hat{s}_{34}^* & \hat{s}_{35}^* & \hat{s}_{36}^* \end{pmatrix} + \mathcal{O}\left(s_{ij}^3\right)$$

$$\varepsilon_{j\alpha} \equiv \frac{\Gamma(N_j \to \ell_\alpha + H) - \Gamma(N_j \to \overline{\ell_\alpha} + \overline{H})}{\sum_\alpha \left[\Gamma(N_j \to \ell_\alpha + H) + \Gamma(N_j \to \overline{\ell_\alpha} + \overline{H})\right]}$$

$$D_j \equiv \left|R_{ej}\right|^2 + \left|R_{\mu j}\right|^2 + \left|R_{\tau j}\right|^2 \quad (j = 4, 5, 6)$$

$$\simeq \frac{1}{8\pi\langle H\rangle^2 D_j} \sum_{j'=4}^{6} \left\{ M_{j'}^2 \, \mathrm{Im}\left[\left(R_{\alpha j}^* R_{\alpha j'}\right) \sum_\beta \left[\left(R_{\beta j}^* R_{\beta j'}\right) \xi(x_{j'j}) + \left(R_{\beta j} R_{\beta j'}^*\right) \zeta(x_{j'j}) \right] \right] \right\}$$

Four types of rephasing invariants in seesaw

★ Given the seesaw mechanism and its weak charged-current interactions, one may define *4* types of rephasing invariants of CP violation for light and heavy Majorana neutrinos:

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{\begin{pmatrix} e & \mu & \tau \end{pmatrix}_{\rm L}} \gamma^\mu \left[U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} + R \begin{pmatrix} N_4 \\ N_5 \\ N_6 \end{pmatrix}_{\rm L} \right] W^-_\mu + {\rm h.c.}$$

oscillations, LNV ← light (U) heavy (R) → collider, LNV, LFV

active and light neutrinos

- ♦ Jarlskog-like invariants for CPV in LFV + LNV cases: $\mathcal{J}^{ii'}_{\alpha\beta} \equiv {\rm Im}\left(U_{\alpha i} U_{\beta i'} U^*_{\alpha i'} U^*_{\beta i}\right)$ √
- ♦ Jarlskog-like invariants for CPV in the LNV processes: $\mathcal{V}^{ii'}_{\alpha\beta} \equiv {\rm Im}\left(U_{\alpha i} U_{\beta i} U^*_{\alpha i'} U^*_{\beta i'}\right)$?

sterile and heavy neutrinos

- ♦ Jarlskog-like invariants for CPV in LFV + LNV cases: $\mathcal{X}^{jj'}_{\alpha\beta} \equiv {\rm Im}\left(R_{\alpha j} R_{\beta j'} R^*_{\alpha j'} R^*_{\beta j}\right)$ √
- ♦ Jarlskog-like invariants for CPV in the LNV processes: $\mathcal{Z}^{jj'}_{\alpha\beta} \equiv {\rm Im}\left(R_{\alpha j} R_{\beta j} R^*_{\alpha j'} R^*_{\beta j'}\right)$ √

Analytical results in the Euler-like parametrization

★ **The invariants of CP violation for heavy neutrinos associated with the LFV and LNV processes:**

$\mathcal{X}^{45}_{e\mu} = s_{14}s_{15}s_{24}s_{25}\sin(\alpha_2-\alpha_1)$	$\mathcal{Z}^{45}_{e\mu} = -s_{14}s_{15}s_{24}s_{25}\sin(\alpha_1+\alpha_2)$	$\mathcal{Z}^{45}_{ee} = -s^2_{14}s^2_{15}\sin 2\alpha_1$
$\mathcal{X}^{45}_{e\tau} = s_{14}s_{15}s_{34}s_{35}\sin(\alpha_3-\alpha_1)$	$\mathcal{Z}^{45}_{e\tau} = -s_{14}s_{15}s_{34}s_{35}\sin(\alpha_1+\alpha_3)$	$\mathcal{Z}^{45}_{\mu\mu} = -s^2_{24}s^2_{25}\sin 2\alpha_2$
$\mathcal{X}^{45}_{\mu\tau} = s_{24}s_{25}s_{34}s_{35}\sin(\alpha_3-\alpha_2)$	$\mathcal{Z}^{45}_{\mu\tau} = -s_{24}s_{25}s_{34}s_{35}\sin(\alpha_2+\alpha_3)$	$\mathcal{Z}^{45}_{\tau\tau} = -s^2_{34}s^2_{35}\sin 2\alpha_3$
$\mathcal{X}^{46}_{e\mu} = s_{14}s_{16}s_{24}s_{26}\sin(\gamma_1-\gamma_2)$	$\mathcal{Z}^{46}_{e\mu} = -s_{14}s_{16}s_{24}s_{26}\sin(\gamma_1+\gamma_2)$	$\mathcal{Z}^{46}_{ee} = +s^2_{14}s^2_{16}\sin 2\gamma_1$
$\mathcal{X}^{46}_{e\tau} = s_{14}s_{16}s_{34}s_{36}\sin(\gamma_1-\gamma_3)$	$\mathcal{Z}^{46}_{e\tau} = -s_{14}s_{16}s_{34}s_{36}\sin(\gamma_1+\gamma_3)$	$\mathcal{Z}^{46}_{\mu\mu} = +s^2_{24}s^2_{26}\sin 2\gamma_2$
$\mathcal{X}^{46}_{\mu\tau} = s_{24}s_{26}s_{34}s_{36}\sin(\gamma_2-\gamma_3)$	$\mathcal{Z}^{46}_{\mu\tau} = -s_{24}s_{26}s_{34}s_{36}\sin(\gamma_2+\gamma_3)$	$\mathcal{Z}^{46}_{\tau\tau} = +s^2_{34}s^2_{36}\sin 2\gamma_3$
$\mathcal{X}^{56}_{e\mu} = s_{15}s_{16}s_{25}s_{26}\sin(\beta_2-\beta_1)$	$\mathcal{Z}^{56}_{e\mu} = -s_{15}s_{16}s_{25}s_{26}\sin(\beta_1+\beta_2)$	$\mathcal{Z}^{56}_{ee} = -s^2_{15}s^2_{16}\sin 2\beta_1$
$\mathcal{X}^{56}_{e\tau} = s_{15}s_{16}s_{35}s_{36}\sin(\beta_3-\beta_1)$	$\mathcal{Z}^{56}_{e\tau} = -s_{15}s_{16}s_{35}s_{36}\sin(\beta_1+\beta_3)$	$\mathcal{Z}^{56}_{\mu\mu} = -s^2_{25}s^2_{26}\sin 2\beta_2$
$\mathcal{X}^{56}_{\mu\tau} = s_{25}s_{26}s_{35}s_{36}\sin(\beta_3-\beta_2)$	$\mathcal{Z}^{56}_{\mu\tau} = -s_{25}s_{26}s_{35}s_{36}\sin(\beta_2+\beta_3)$	$\mathcal{Z}^{56}_{\tau\tau} = -s^2_{35}s^2_{36}\sin 2\beta_3$

Phases: $\alpha_i \equiv \delta_{i4}-\delta_{i5}\,, \quad \beta_i \equiv \delta_{i5}-\delta_{i6}\,, \quad \gamma_i \equiv \delta_{i6}-\delta_{i4}$

Totally six independent CP-violating phases.

with $\alpha_i+\beta_i+\gamma_i = 0$ for $i = 1,2,3.$

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CP asymmetries of heavy neutrino decays

★ **Flavor-dependent CPV:**

$$\varepsilon_{j\alpha} \simeq \frac{-1}{8\pi\langle H\rangle^2 D_j} \sum_{j'=4}^{6} \left[M_{j'}^2 \sum_\beta \left[\mathcal{Z}_{\alpha\beta}^{jj'} \xi(x_{j'j}) + \mathcal{X}_{\alpha\beta}^{jj'} \zeta(x_{j'j}) \right] \right]$$

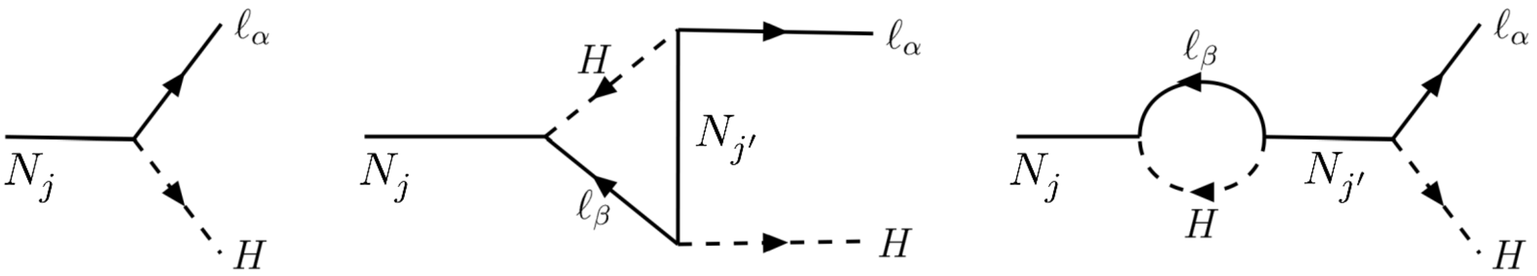
$$\varepsilon_j \equiv \varepsilon_{je} + \varepsilon_{j\mu} + \varepsilon_{j\tau}$$

★ **Flavor-independent CPV:**

$$\varepsilon_j \simeq \frac{-1}{8\pi\langle H\rangle^2 D_j} \sum_{j'=4}^{6} \left[M_{j'}^2 \left(\sum_\alpha \mathcal{Z}_{\alpha\alpha}^{jj'} + 2\sum_{\alpha<\beta} \mathcal{Z}_{\alpha\beta}^{jj'} \right) \xi(x_{j'j}) \right]$$

The loop functions:

$$\xi(x_{j'j}) = \sqrt{x_{j'j}} \left\{ 1 + 1/\left(1 - x_{j'j}\right) + \left(1 + x_{j'j}\right) \ln\left[x_{j'j}/\left(1 + x_{j'j}\right)\right] \right\} \text{ and } \zeta(x_{j'j}) = 1/\left(1 - x_{j'j}\right)$$



$$x_{j'j} \equiv \frac{M_{j'}^2}{M_j^2}$$

Leptogenesis

Flavor oscillations of active neutrinos

★ **Flavor oscillations of active neutrinos in the seesaw mechanism with slight PMNS non-unitarity:**

$$P(\nu_\alpha \to \nu_\beta) = \frac{1}{D_{\alpha\beta}} \left[\sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2 \sum_{i<i'} \mathrm{Re} \left(U_{\alpha i} U_{\beta i'} U^*_{\alpha i'} U^*_{\beta i} \right) \cos \Delta_{i'i} + 2 \sum_{i<i'} \mathcal{J}^{ii'}_{\alpha\beta} \sin \Delta_{i'i} \right]$$

$$P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = \frac{1}{D_{\alpha\beta}} \left[\sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2 \sum_{i<i'} \mathrm{Re} \left(U_{\alpha i} U_{\beta i'} U^*_{\alpha i'} U^*_{\beta i} \right) \cos \Delta_{i'i} - 2 \sum_{i<i'} \mathcal{J}^{ii'}_{\alpha\beta} \sin \Delta_{i'i} \right]$$

where $D_{\alpha\beta} \equiv \left(UU^\dagger\right)_{\alpha\alpha} \left(UU^\dagger\right)_{\beta\beta} = \left(AA^\dagger\right)_{\alpha\alpha} \left(AA^\dagger\right)_{\beta\beta}$, $\Delta_{i'i} \equiv \Delta m^2_{i'i} L / (2E)$ with $\Delta m^2_{i'i} \equiv m^2_{i'} - m^2_i$

CP violating asymmetries: $\mathcal{A}_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) - P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = \dfrac{4}{D_{\alpha\beta}} \displaystyle\sum_{i<i'} \mathcal{J}^{ii'}_{\alpha\beta} \sin \Delta_{i'i}$

Example for a LBL case in vacuum:

$$\mathcal{A}_{\mu e} \simeq \boxed{-16 \mathcal{J}_\nu \sin \frac{\Delta_{21}}{2} \sin \frac{\Delta_{31}}{2} \sin \frac{\Delta_{32}}{2}} - 4 c_{13} \Big\{ c_{12} s_{12} c_{23} \mathrm{Im} \left(\underline{a_{21}} e^{-\mathrm{i}\delta_{21}} \right) \sin \Delta_{21} + c^2_{12} s_{13} s_{23} \mathrm{Im} \left[\underline{a_{21}} e^{\mathrm{i}\left(\delta_{23} - \delta_{13}\right)} \right] \sin \Delta_{31} + s^2_{12} s_{13} s_{23} \mathrm{Im} \left[\underline{a_{21}} e^{\mathrm{i}\left(\delta_{23} - \delta_{13}\right)} \right] \sin \Delta_{32} \Big\}$$

Terrestrial matter effects are entangled with the PMNS non-unitarity effects (e.g., Y.F. Li, ZZX, J.Y. Zhu 2019).

$$a_{21} \equiv \hat{s}^*_{24} \hat{s}_{14} + \hat{s}^*_{25} \hat{s}_{15} + \hat{s}^*_{26} \hat{s}_{16}$$

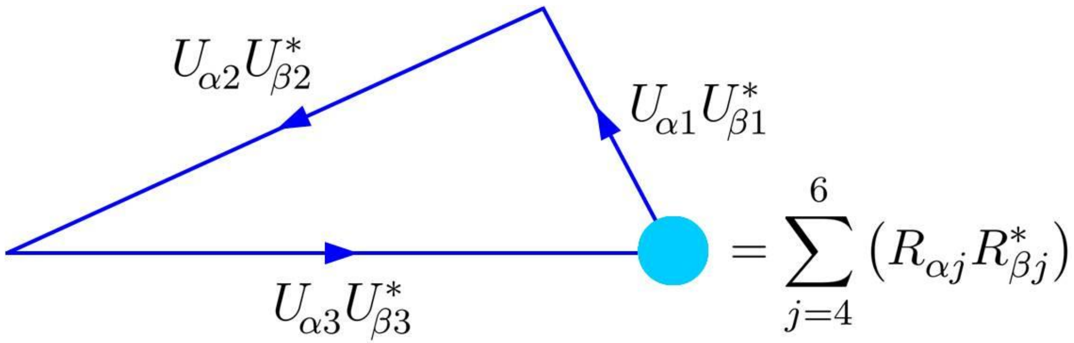
The PMNS unitarity polygons in the seesaw case

♦ One can show that the leading terms of all these Jarlskog-like invariants are the same, coming from the unitarity limit of the PMNS matrix (ZZX, D. Zhang, 2009.09717):

$$\mathcal{J}^{ii'}_{\alpha\beta} \equiv \mathrm{Im}\left(U_{\alpha i} U_{\beta i'} U^*_{\alpha i'} U^*_{\beta i}\right)$$

$$\mathcal{J}^{ii'}_{\alpha\beta} = \mathcal{J}_\nu + \text{corrections}$$

$\mathcal{J}_\nu$: $\leq$ 3% corrections: $<$ 0.1%



$$\mathcal{J}^{12}_{e\mu} \simeq \mathcal{J}_\nu + c_{12} s_{12} c_{23} \mathrm{Im} a_{21}$$

$$\mathcal{J}^{12}_{\tau e} \simeq \mathcal{J}_\nu + c_{12} s_{12} s_{23} \mathrm{Im} a_{31}$$

$$\mathcal{J}^{12}_{\mu\tau} \simeq \mathcal{J}_\nu + c_{12} s_{12} c_{23} s_{23} \left(s_{23} \mathrm{Im} a_{21} + c_{23} \mathrm{Im} a_{31}\right)$$

$$\mathcal{J}^{23}_{\mu\tau} \simeq \mathcal{J}_\nu + c_{12} c_{23} s_{23} \left(s_{12} s_{23} \mathrm{Im} a_{21} + s_{12} c_{23} \mathrm{Im} a_{31} + c_{12} \mathrm{Im} a_{32}\right)$$

$$\mathcal{J}^{31}_{\mu\tau} \simeq \mathcal{J}_\nu + s_{12} c_{23} s_{23} \left(c_{12} s_{23} \mathrm{Im} a_{21} + c_{12} c_{23} \mathrm{Im} a_{31} - s_{12} \mathrm{Im} a_{32}\right)$$

$$\mathcal{J}^{23}_{e\mu} \simeq \mathcal{J}^{31}_{e\mu} \simeq \mathcal{J}^{23}_{\tau e} \simeq \mathcal{J}^{31}_{\tau e} \simeq \mathcal{J}_\nu$$

$$a_{ii'} \equiv \hat{s}^*_{i4} \hat{s}_{i'4} + \hat{s}^*_{i5} \hat{s}_{i'5} + \hat{s}^*_{i6} \hat{s}_{i'6}$$

So it is absolutely safe to neglect the tiny non-unitarity effects on CPV in neutrino oscillations.

Life is much easier in the minimal seesaw case

♦ **The Jarlskog invariant and the CP-violating asymmetries of heavy Majorana neutrinos depend on 3 original CP phases in a relatively simple way if the PMNS non-unitarity is neglected (ZZX 2023):**

	$\sin 2\alpha_1$	$\sin 2\alpha_2$	$\sin 2\alpha_3$	$\sin(\alpha_1+\alpha_2)$	$\sin(\alpha_1+\alpha_3)$	$\sin(\alpha_2+\alpha_3)$	$\sin(\alpha_1-\alpha_2)$	$\sin(\alpha_2-\alpha_3)$	$\sin(\alpha_3-\alpha_1)$
$\mathcal{J}_\nu$	√	√	√	√	√	√	√	√	√
ε_{4e}	√			√	√		√		√
$\varepsilon_{4\mu}$		√		√		√	√	√	
$\varepsilon_{4\tau}$			√		√	√		√	√
ε_4	√	√	√	√	√	√			
ε_{5e}	√			√	√		√		√
$\varepsilon_{5\mu}$		√		√		√	√	√	
$\varepsilon_{5\tau}$			√		√	√		√	√
ε_5	√	√	√	√	√	√			

♦ **Parameters in the minimal (3 + 2) seesaw: 7 = 2 + 3 + 2 (low) ↔ 11 = 2 + 6 + 3 (high).**

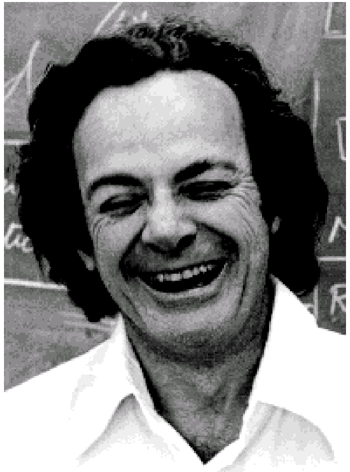
♦ **It is also very important to calculate the neutrino mass-squared differences, active flavor mixing angles and all LNV and LFV effects at low energies with the *original* seesaw parameters/invariants in the seesaw framework (ZZX, J.Y. Zhu, 2412.17698).**

Concluding remarks

★ Question: rephasing invariants (e.g., moduli of the PMNS matrix elements, the Jarlskog invariant or its analogs, angles of the unitarity triangles or polygons) and basis-dependent parametrizations, which set is more useful in the studies of neutrino physics?

★ My personal answer: they are two sides of the same coin, and one of them may be more useful in making the underlying physics more transparent, or making correlative relations between intrinsic model parameters and observable quantities more straightforward.

Theories of the known, which are described by different physical ideas, may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made. I, therefore, think that a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory available to him —— R.P. Feynman's Nobel Lecture.



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