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# All-loop equations for the renormalization group functions of supersymmetric theories

Sometimes the structure of the surrounding world can be understood by studying quantum corrections. For instance, the very precise agreement of the theoretical prediction of the electron anomalous magnetic moment with the experimental data tells us that the nature is described by quantum field theory.
The unification of running couplings and absence of quadratically divergent quantum corrections to the Higgs boson mass may be considered as indirect indications to the existence of supersymmetry and Grand Unification.

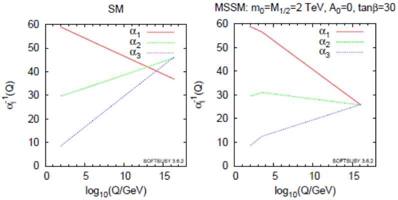


Figure 94.1: Running couplings in SM and MSSM using two-loop RG evolution. The SUSY threshold at 2 TeV is clearly visible on the MSSM side. (We thank Ben Allanach for providing the plots created using SOFTSUSY [61].)

Some important information about new physics can be obtained from the detailed analysis of quantum corrections to the (lightest) Higgs boson mass in supersymmetric theories, anomalous magnetic moment of muon, etc.

It is convenient to describe $\mathcal{N}=1$ supersymmetric gauge theories in $\mathcal{N}=1$ superspace. Then, for the renormalizable theories, the action can be written as

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\, d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mbox{c.c.}\Big\}.$$

Here $V$ is the gauge superfield, $\phi_i$ are the chiral matter superfields in the representation $R$ of the gauge group $G$, and

$$W_a = \frac{1}{8}\bar D^2\Big(e^{-2V}D_a e^{2V}\Big)$$

is the supersymmetric gauge field strength.

The gauge invariant theory is obtained if the Yukawa couplings and masses satisfy the constraints

$$m_0^{im}(T^A)_m{}^j + m_0^{mj}(T^A)_m{}^i = 0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k + \lambda_0^{imk}(T^A)_m{}^j + \lambda_0^{mjk}(T^A)_m{}^i = 0,$$

where $(T^A)_i{}^j$ are the generators of the gauge group $G$ in the representation $R$.

# The NSVZ $\beta$-function for $\mathcal{N}=1$ supersymmetric theories

In $\mathcal{N}=1$ supersymmetric theories there is an all-loop relation between the $\beta$-function and the anomalous dimension of the matter superfields called the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) $\beta$-function

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381; Phys. Lett. **166B**(1986), 329;
D. R. T. Jones, Phys. Lett. **123B** (1983), 45;
M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986), 456.

For a general $\mathcal{N}=1$ supersymmetric gauge theory with a single gauge coupling it can be written in the form

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here $\alpha$ and $\lambda$ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

Nevertheless, it is necessary to remember that the NSVZ equation is valid only for certain renormalization prescriptions.

Note that in the $\overline{\text{DR}}$ scheme the NSVZ equation is not valid starting from the order $O(\alpha^4)$ (the three-loop approximation for the $\beta$-function and the two-loop approximation for the anomalous dimension)

I. Jack, D. R. T. Jones and C. G. North, Phys.Lett. B **386** (1996) 138; Nucl.Phys. B **486** (1997) 479;
R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila and M. Steinhauser, JHEP **0612** (2006) 024.

However, in this case it is possible to make a special redefinition of the coupling constant which restores the NSVZ relation.

The all-loop NSVZ schemes have been constructed with the help of the higher covariant derivative regularization

A. A. Slavnov, Nucl. Phys. B **31** (1971), 301;
Theor. Math. Phys. **13** (1972), 1064; **33** (1977), 977.

# The HD+MSL scheme

In the supersymmetric case the higher covariant derivative regularization can be formulated in terms of superfields and, therefore, does not break supersymmetry

V. K. Krivoshchekov, Theor. Math. Phys. **36** (1978), 745;
P. C. West, Nucl. Phys. B **268** (1986), 113.

In this case (logarithmic) divergences are given by powers of $\ln\Lambda/\mu$, where $\Lambda$ is a dimensionful regularization parameter, and $\mu$ is the renormalization point.

The NSVZ $\beta$-function is valid in all loops if a supersymmetric theory is regularized by Higher covariant Derivatives and the renormalization is made by Minimal Subtractions of Logarithms (the so-called HD+MSL scheme), see

K.S., Eur. Phys. J. C **80** (2020) no.10, 911

and references therein.

Minimal subtractions of logarithms mean that only powers of $\ln\Lambda/\mu$ (where $\Lambda$ is the cutoff parameter and $\mu$ is the renormalization point) are included into renormalization constants,

A. L. Kataev, K.S., Nucl. Phys. B **875** (2013), 459.

# Theories with multiple gauge couplings

In what follows we will mostly investigate theories with multiple gauge couplings. In this case the gauge group is a direct product

$$G = G_1 \times G_2 \times \ldots \times G_n,$$

where any $G_i$ is either a simple group or $U(1)$. In this case there are $n$ gauge coupling constants $\alpha_1$, $\alpha_2$, ..., $\alpha_n$.
Such theories can be interesting for phenomenology because they include

- QCD+QED
- The Standard Model
- The MSSM
- Some Grand Unified Theories, e.g., the flipped $SU(5)$ theory.

We argue that in some $\mathcal{N}=1$ supersymmetric theories with multiple couplings the couplings do not run independently, and (sometimes) it is possible to construct all-loop renormalization group invariants (RGIs) from the gauge and Yukawa couplings. We will also discuss under what renormalization prescriptions the renormalization group invariance of the corresponding expressions is valid in all orders.

The simplest example of a theory with two gauge coupling constants $\alpha_s \equiv g^2/4\pi$ and $\alpha = e^2/4\pi$ is QCD+QED. In the massless limit this theory is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\, F_{\mu\nu}^2 - \frac{1}{4e^2}\boldsymbol{F}_{\mu\nu}^2 + \sum_{\mathsf{a}=1}^{N_f} i\bar\psi_{\mathsf{a}}\gamma^\mu \mathcal{D}_\mu \psi_{\mathsf{a}},$$

which is invariant under the transformations of the gauge group $G\times U(1)$. The Dirac spinors $\psi_{\mathsf{a}}$ (where the subscript a numerates flavors) lie in a certain irreducible representation $R$ of the group $G$ and have the electromagnetic charges $q_{\mathsf{a}}$. In this case the covariant derivatives are written in the form

$$\mathcal{D}_\mu\psi_{\mathsf{a}} = \partial_\mu\psi_{\mathsf{a}} + iA_\mu\psi_{\mathsf{a}} + iq_{\mathsf{a}}\boldsymbol{A}_\mu\psi_{\mathsf{a}},$$

where $A_\mu$ and $\boldsymbol{A}_\mu$ are the non-Abelian and Abelian gauge fields, respectively. The corresponding gauge field strengths are given by the expressions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]; \qquad \boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu.$$

In quantum field theory the couplings $\alpha_s$ and $\alpha$ depend on scale,

$$\frac{d\alpha}{d\ln\mu} = \beta(\alpha,\alpha_s); \qquad \frac{d\alpha_s}{d\ln\mu} = \beta_s(\alpha_s,\alpha).$$

It is convenient to formulate the supersymmetric version of the above model in terms of superfields

$$S=\frac{1}{2g^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a+\frac{1}{4e^2}\,\mbox{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{\mathsf{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathsf{a}}^+e^{2V+2q_{\mathsf{a}}\boldsymbol{V}}\phi_{\mathsf{a}}+\widetilde\phi_{\mathsf{a}}^+e^{-2V^T-2q_{\mathsf{a}}\boldsymbol{V}}\widetilde\phi_{\mathsf{a}}\Big),$$

because in this case $\mathcal{N}=1$ supersymmetry is manifest. Here $V$ and $\boldsymbol{V}$ are the gauge superfields corresponding to the subgroups $G$ and $U(1)$, respectively. The chiral matter superfields $\phi_{\mathsf{a}}$ and $\widetilde\phi_{\mathsf{a}}$ belong to the (conjugated) representations $R$ and $\overline{R}$, respectively, and have opposite $U(1)$ charges.

Two supersymmetric gauge superfield strengths are written in the form

$$W_a=\frac{1}{8}\bar D^2\Big(e^{-2V}D_a e^{2V}\Big);\qquad \boldsymbol{W}_a=\frac{1}{4}\bar D^2 D_a\boldsymbol{V}.$$

Is it possible to relate running of two gauge coupling constants in this model?

A. L. Kataev, K.S., JETP Lett. **121** (2025) no.5, 315.

The NSVZ equations can also be written for theories with multiple gauge couplings,

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761;
D. Korneev, D. Plotnikov, K.S. and N. Tereshina, JHEP **10** (2021), 046.

In the particular case $q_{\sf a} = 1$ for $\mathcal{N} = 1$ SQCD+SQED they have the form

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi(1 - C_2\alpha_s/2\pi)}\Big[3C_2 - 2T(R)N_f\Big(1 - \gamma(\alpha_s,\alpha)\Big)\Big];$$

$$\frac{\beta(\alpha,\alpha_s)}{\alpha^2} = \frac{1}{\pi}\,\mbox{dim}\,R\,N_f\Big(1 - \gamma(\alpha_s,\alpha)\Big),$$

where we took into account that if the representation for the matter superfields is irreducible, then in the case under consideration

$$\gamma(\alpha_s,\alpha)_i{}^j = \gamma(\alpha_s,\alpha)\cdot\delta_i^j,$$

where $i$ and $j$ include both the indices numerating chiral matter superfields $\phi_{\sf a}$ and $\widetilde\phi_{\sf a}$ and the indices corresponding to the representation $R$ (or $\overline{R}$).

Comparing the above expressions for the $\beta$-functions we see that the anomalous dimension of the matter superfields can be eliminated.

After eliminating the anomalous dimension of the matter superfields we obtain that the $\beta$-functions satisfy the all-order exact equation

$$\Big(1-\frac{C_2\alpha_s}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{3C_2}{2\pi} + \frac{T(R)}{\mbox{dim}\, R}\cdot\frac{\beta(\alpha,\alpha_s)}{\alpha^2}.$$

Evidently, this equation is valid in the HD+MSL scheme, because the original NSVZ equations are satisfied for this renormalization prescription.

Taking into account the boundary conditions for the HD+MSL scheme, it is possible to integrate the relation between the $\beta$-functions over $\mu$. Then we obtain the equation which relates running of the strong and electromagnetic couplings in the theory under consideration.

$$\frac{1}{\alpha_s} - \frac{1}{\alpha_{s0}} + \frac{C_2}{2\pi}\ln\frac{\alpha_s}{\alpha_{0s}} = -\frac{3C_2}{2\pi}\ln\frac{\Lambda}{\mu} + \frac{T(R)}{\mbox{dim}\, R}\Big(\frac{1}{\alpha}-\frac{1}{\alpha_0}\Big).$$

This in particular implies that the expression

$$\Big(\frac{\alpha_s}{\mu^3}\Big)^{C_2}\exp\Big(\frac{2\pi}{\alpha_s} - \frac{T(R)}{\mbox{dim}\, R}\cdot\frac{2\pi}{\alpha}\Big) = \mbox{RGI}$$

is the renormalization group invariant, i.e. the expression which vanishes afer differentiating with respect to $\ln\mu$.

The three-loop verification of this general result has been done in

O. Haneychuk, K. S., Eur. Phys. J. C **85** (2025) no.5, 540.

It is important that in this approximation the dependence on regularization and renormalization parameters becomes essential. We will use a renormalization prescription for that

$$\frac{1}{\alpha_0} = \frac{1}{\alpha} - \frac{N_f\,\textsf{dim}\,R}{\pi}\Big(\ln\frac{\Lambda}{\mu} + d_1\Big) - \frac{\alpha_s}{\pi^2}N_f C(R)\,\textsf{dim}\,R\Big(\ln\frac{\Lambda}{\mu} + d_2\Big) - \frac{\alpha}{\pi^2}N_f\,\textsf{dim}\,R \times\Big(\ln\frac{\Lambda}{\mu} + \widetilde{d}_2\Big) + O(\alpha_s^2, \alpha_s\alpha, \alpha^2);$$

$$\frac{1}{\alpha_{s0}} = \frac{1}{\alpha_s} + \frac{3C_2}{2\pi}\Big(\ln\frac{\Lambda}{\mu} + b_{11}\Big) - \frac{N_f T(R)}{\pi}\Big(\ln\frac{\Lambda}{\mu} + b_{12}\Big) + \frac{3\alpha_s}{4\pi^2}(C_2)^2\Big(\ln\frac{\Lambda}{\mu} + b_{21}\Big) - \frac{\alpha_s}{2\pi^2}N_f C_2 T(R)\Big(\ln\frac{\Lambda}{\mu} + b_{22}\Big) - \frac{\alpha_s}{\pi^2}N_f C(R)T(R)\Big(\ln\frac{\Lambda}{\mu} + b_{23}\Big) - \frac{\alpha}{\pi^2}N_f T(R)\Big(\ln\frac{\Lambda}{\mu} + \widetilde{b}_{21}\Big) + O(\alpha_s^2, \alpha_s\alpha, \alpha^2).$$

The HD+MSL scheme is obtained if $b_{11} = b_{12} = b_{21} = b_{22} = b_{23} = \widetilde{b}_{21} = 0$, $d_1 = d_2 = \widetilde{d}_2 = 0$. The regularization parameters essential in the three-loop approximation are

$$A \equiv \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\frac{1}{R(x)};\qquad a_\varphi \equiv \frac{M_\varphi}{\Lambda};\qquad a_G \equiv \frac{M_G}{\Lambda};\qquad a_1 \equiv \frac{M_1}{\Lambda}.$$

# The three-loop RGFs for $\mathcal{N}=1$ SQCD+SQED

The three-loop $\beta$-functions have the form

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2} = \frac{N_f\,\textbf{dim}\,R}{\pi}\Big\{1+\frac{\alpha}{\pi}+\frac{\alpha_s}{\pi}C(R)-\frac{1}{2\pi^2}\Big(\alpha+\alpha_s C(R)\Big)^2-\frac{\alpha^2}{\pi^2}N_f\,\textbf{dim}\,R\Big(\ln a_1 +1+\frac{A}{2}+\widetilde{d}_2-d_1\Big)+\frac{3\alpha_s^2}{2\pi^2}C_2 C(R)\Big(\ln a_\varphi+1+\frac{A}{2}+d_2-b_{11}\Big)-\frac{\alpha_s^2}{\pi^2}N_f C(R)T(R)\times\Big(\ln a_G+1+\frac{A}{2}+d_2-b_{12}\Big)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi}\Big(3C_2-2N_f T(R)\Big)+\frac{\alpha}{\pi^2}N_f T(R)+\frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2+2N_f C_2 T(R)+4N_f C(R)T(R)\Big)-\frac{\alpha^2}{\pi^3}(N_f)^2 T(R)\,\textbf{dim}\,R\Big(\ln a_1+1+\frac{A}{2}+\widetilde{b}_{21}-d_1\Big)-\frac{1}{2\pi^3}N_f T(R)\times\Big(\alpha+\alpha_s C(R)\Big)^2+\frac{\alpha\alpha_s}{2\pi^3}N_f C_2 T(R)-\frac{3\alpha_s^2}{8\pi^3}(C_2)^3\Big(1+3b_{21}-3b_{11}\Big)+\frac{\alpha_s^2}{4\pi^3}N_f(C_2)^2\times T(R)\Big(1+3b_{21}-3b_{11}+3b_{22}-3b_{12}\Big)+\frac{3\alpha_s^2}{2\pi^3}N_f C_2 C(R)T(R)\Big(\ln a_\varphi+\frac{4}{3}+\frac{A}{2}+b_{23}-b_{11}\Big)-\frac{\alpha_s^2}{2\pi^3}(N_f)^2 C_2 T(R)^2(b_{22}-b_{12})-\frac{\alpha_s^2}{\pi^3}(N_f)^2 C(R)T(R)^2\Big(\ln a_G+1+\frac{A}{2}+b_{23}-b_{12}\Big)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

They satisfy the exact relation if $\widetilde{b}_{21}=\widetilde{d}_2$, $b_{21}=b_{11}$, $b_{22}=b_{12}$, $b_{23}=d_2$. In particular, this relation is valid in the HD+MSL scheme.

# The three-loop results for the $\overline{\text{DR}}$ scheme

The three-loop $\beta$-functions in the $\overline{\text{DR}}$ scheme are given by

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2}\Big|_{\overline{\text{DR}}} = \frac{N_f\,\text{dim}\,R}{\pi}\Big\{1+\frac{\alpha}{\pi}+\frac{\alpha_s}{\pi}C(R)-\frac{1}{2\pi^2}\Big(\alpha+\alpha_s C(R)\Big)^2-\frac{3\alpha^2}{4\pi^2}N_f\,\text{dim}\,R$$
$$+\frac{9\alpha_s^2}{8\pi^2}C_2C(R)-\frac{3\alpha_s^2}{4\pi^2}N_fC(R)T(R)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}\Big|_{\overline{\text{DR}}} = -\frac{1}{2\pi}\Big(3C_2-2N_fT(R)\Big)+\frac{\alpha}{\pi^2}N_fT(R)+\frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2+2N_fC_2T(R)$$
$$+4N_fC(R)T(R)\Big)-\frac{3\alpha^2}{4\pi^3}(N_f)^2T(R)\,\text{dim}\,R-\frac{1}{2\pi^3}N_fT(R)\Big(\alpha+\alpha_sC(R)\Big)^2+\frac{\alpha\alpha_s}{2\pi^3}N_fC_2$$
$$\times T(R)-\frac{21\alpha_s^2}{32\pi^3}(C_2)^3+\frac{5\alpha_s^2}{8\pi^3}N_f(C_2)^2T(R)+\frac{13\alpha_s^2}{8\pi^3}N_fC_2C(R)T(R)-\frac{\alpha_s^2}{8\pi^3}(N_f)^2C_2T(R)^2$$
$$-\frac{3\alpha_s^2}{4\pi^3}(N_f)^2C(R)T(R)^2+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

In this case the relation between the $\beta$-functions is not satisfied starting from the three-loop approximation,

$$\Big(1-\frac{\alpha_sC_2}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}+\frac{3C_2}{2\pi}-\frac{T(R)}{\text{dim}\,R}\cdot\frac{\beta(\alpha_s,\alpha)}{\alpha^2}=-\frac{9\alpha_s^2}{32\pi^3}(C_2)^3$$
$$+\frac{3\alpha_s^2}{8\pi^3}(C_2)^2N_fT(R)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\neq O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

# The minimal scheme for $\mathcal{N}=1$ SQCD+SQED

The explicit dependence of the $\beta$-functions on the renormalization parameters allows constructing a subtraction scheme in which the relation between the $\beta$-function is valid and RGFs are as simple as possible. In this case

$$b_{12} = b_{11} + \ln\frac{a_G}{a_\varphi};\qquad d_2 = b_{11} - \ln a_\varphi - 1 - \frac{A}{2};\qquad \widetilde{d}_2 = d_1 - \ln a_1 - 1 - \frac{A}{2}.$$

In this case the expressions for the $\beta$-functions take the simplest form

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2} = \frac{N_f\,\textbf{dim}\,R}{\pi}\Big\{1 + \frac{\alpha}{\pi} + \frac{\alpha_s}{\pi}C(R) - \frac{1}{2\pi^2}\Big(\alpha + \alpha_s C(R)\Big)^2 + O(\alpha_s^3, \alpha_s^2\alpha, \alpha_s\alpha^2, \alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \Big(1 + \frac{\alpha_s C_2}{2\pi}\Big)\Big[\frac{\alpha}{\pi^2}N_f T(R) + \frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2 + 2N_f$$
$$\times C_2 T(R) + 4N_f C(R)T(R)\Big)\Big] - \frac{1}{2\pi^3}N_f T(R)\Big(\alpha + \alpha_s C(R)\Big)^2 + O(\alpha_s^3, \alpha_s^2\alpha, \alpha_s\alpha^2, \alpha^3).$$

This scheme is analogous to the minimal scheme for $\mathcal{N}=1$ SQED with $N_f$ flavors, in which only terms without $N_f$ survive in the anomalous dimension and

$$\frac{\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi}(1-\gamma(\alpha)) = \frac{N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2} + \frac{\alpha^3}{2\pi^3} + O(\alpha^4)\Big),$$

I. Shirokov and K.S., JHEP **04** (2022), 108.

# The Minimal Supersymmetric Standard Model (MSSM)

The analogous research for MSSM has been done in

D. Rystsov, K.S., Phys. Rev. D **111** (2025) no.1, 016012.

The MSSM is the simplest supersymmetric extension of the Standard Model. It is a gauge theory with the group $SU_3\times SU_2\times U_1$ and softly broken supersymmetry. Quarks, leptons, and Higgs fields are components of the chiral matter superfields:

| Superfield | $SU_3$ | $SU_2$ | $U_1\ (Y)$ | Superfield | $SU_3$ | $SU_2$ | $U_1\ (Y)$ |
|---|---|---|---|---|---|---|---|
| $3\times Q$ | $\bar{3}$ | 2 | $-1/6$ | $3\times N$ | 1 | 1 | $0$ |
| $3\times U$ | 3 | 1 | $2/3$ | $3\times E$ | 1 | 1 | $-1$ |
| $3\times D$ | 3 | 1 | $-1/3$ | $H_d$ | 1 | 2 | $1/2$ |
| $3\times L$ | 1 | 2 | $1/2$ | $H_u$ | 1 | 2 | $-1/2$ |

where for the superfields which include left quarks and leptons we use the brief notations

$$Q=\begin{pmatrix}\widetilde{U}\\ \widetilde{D}\end{pmatrix};\qquad L=\begin{pmatrix}\widetilde{N}\\ \widetilde{E}\end{pmatrix}.$$

# The MSSM couplings and superpotential

The MSSM contains three gauge couplings

$$\alpha_3 = \frac{e_3^2}{4\pi}; \qquad \alpha_2 = \frac{e_2^2}{4\pi}; \qquad \alpha_1 = \frac{5}{3}\cdot\frac{e_1^2}{4\pi}$$

corresponding to the subgroups $SU(3)$, $SU(2)$, and $U(1)$, respectively. (The factor $5/3$ in the coupling constant $\alpha_1$ is introduced in order that the unification of couplings has the form $\alpha_1 = \alpha_2 = \alpha_3$.) There are also the dimensionless Yukawa couplings $(Y_U)_{IJ}$, $(Y_D)_{IJ}$, and $(Y_E)_{IJ}$ (which are $3\times 3$ matrices) inside the superpotential

$$\begin{aligned} W = (Y_U)_{IJ}\left(\widetilde{U}\ \widetilde{D}\right)^a_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{u1} \\ H_{u2} \end{pmatrix} U_{aJ} \\ +(Y_D)_{IJ}\left(\widetilde{U}\ \widetilde{D}\right)^a_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} D_{aJ} \\ +(Y_E)_{IJ}\left(\widetilde{N}\ \widetilde{E}\right)_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} E_J \\ +\boldsymbol{\mu}\left(H_{u1}\ H_{u2}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix}. \end{aligned}$$

Moreover, the superpotential includes a term with the parameter $\boldsymbol{\mu}$, which has the dimension of mass.

# The NSVZ equations for the MSSM

The renormalization group running of the gauge couplings in the MSSM can be described exactly in all loops with the help of the NSVZ $\beta$-functions

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761,

$$\frac{\beta_1}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11 + \mbox{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$

$$\frac{\beta_2}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1 + \mbox{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$

$$\frac{\beta_3}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3 + \mbox{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big)\Big].$$

They relate three gauge $\beta$-functions of the theory to the anomalous dimensions of the chiral matter superfields. The renormalization group functions (RGFs) are defined by the equations

$$\beta_i(\alpha, Y) = \frac{d\alpha_i}{d\ln\mu}\Big|_{\alpha_0, Y_0=\mbox{const}};\qquad \gamma_i(\alpha, Y) = \frac{d\ln Z_i}{d\ln\mu}\Big|_{\alpha_0, Y_0=\mbox{const}},$$

where the subscript $0$ denotes the bare values.

# The exact equations describing the renormalization of the MSSM Yukawa couplings

RGFs describing the renormalization of the Yukawa couplings and of the parameter $\boldsymbol{\mu}$ can also be related to the anomalous dimensions of the matter superfields due to the nonrenormalization of the superpotential

M. T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B **159** (1979), 429.

$$\frac{dY_U}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} Y_U + (\gamma_Q)^T Y_U + Y_U \gamma_U\Big);$$

$$\frac{dY_D}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_D + (\gamma_Q)^T Y_D + Y_D \gamma_D\Big);$$

$$\frac{dY_E}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_E + (\gamma_L)^T Y_E + Y_E \gamma_E\Big);$$

$$\frac{d\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big)\boldsymbol{\mu}.$$

It is important that these equations are valid in the HD+MSL scheme because in this scheme all renormalization constants contain only powers of $\ln\Lambda/\mu$, where $\Lambda$ is the dimensionful regularization parameter.

# The equations for the determinants of the Yukawa matrices

The renormalization group equations for the Yukawa couplings can be multiplied by the corresponding inverse matrices. After that, it is possible to calculate traces of the resulting equations using the formula

$$\mbox{tr}\Big[M^{-1}\frac{dM}{d\ln\mu}\Big] = \frac{d}{d\ln\mu}\mbox{tr}\ln M = \frac{d}{d\ln\mu}\ln\det M.$$

Then (taking into account that the indices numerating generations range from 1 to 3), we see that the equations describing how the determinants of the Yukawa matrices depend on the renormalization point $\mu$ are written as

$$\gamma_{\det Y_U} \equiv \frac{d\ln\det Y_U}{d\ln\mu} = \mbox{tr}\Big[(Y_U)^{-1}\frac{dY_U}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_u} + \mbox{tr}\big(\gamma_Q + \gamma_U\big)\Big);$$

$$\gamma_{\det Y_D} \equiv \frac{d\ln\det Y_D}{d\ln\mu} = \mbox{tr}\Big[(Y_D)^{-1}\frac{dY_D}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}\big(\gamma_Q + \gamma_D\big)\Big);$$

$$\gamma_{\det Y_E} \equiv \frac{d\ln\det Y_E}{d\ln\mu} = \mbox{tr}\Big[(Y_E)^{-1}\frac{dY_E}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}\big(\gamma_L + \gamma_E\big)\Big).$$

They can be solved together with the NSVZ equations and the equation describing the renormalization of the parameter $\boldsymbol{\mu}$.

# The renormalization group equations for the (rigid part of the) MSSM

Collecting the above equations we obtain the system of differential equations describing the renormalization of the MSSM parameters exactly in all orders

$$\frac{d}{d\ln\mu}\Big(\frac{5}{3}\cdot\frac{2\pi}{\alpha_1}\Big) = -11 + \mbox{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_2} + 2\ln\alpha_2\Big) = -1 + \mbox{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3} + 3\ln\alpha_3\Big) = 3 + \mbox{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big);$$

$$\frac{d\ln\det Y_U}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_u} + \mbox{tr}(\gamma_Q + \gamma_U)\Big);$$

$$\frac{d\ln\det Y_D}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}(\gamma_Q + \gamma_D)\Big);$$

$$\frac{d\ln\det Y_E}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}(\gamma_L + \gamma_E)\Big);$$

$$\frac{d\ln\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big).$$

The anomalous dimensions of the chiral matter superfields and $\boldsymbol{\mu}$ can be eliminated, thereby obtaining a differential equation which contains only derivatives of the gauge and Yukawa couplings.

Eliminating the anomalous dimensions of the matter superfields, it is possible to obtain the equations

$$0=\Big(\frac{1}{\alpha_2}-\frac{\pi}{\alpha_2^2}\Big)\beta_2-\frac{5\pi}{3{\alpha_1}^2}\beta_1+6+3\gamma_{\boldsymbol{\mu}}-\gamma_{\det Y_E}-\frac{4}{3}\gamma_{\det Y_U}-\frac{1}{3}\gamma_{\det Y_D};$$

$$0=\Big(\frac{3}{\alpha_3}-\frac{2\pi}{\alpha_3^2}\Big)\beta_3-3+3\gamma_{\boldsymbol{\mu}}-\gamma_{\det Y_U}-\gamma_{\det Y_D}.$$

Integrating them with respect to $\ln\mu$ we obtain two independent RGIs

$$\text{RGI}_1=\frac{\boldsymbol{\mu}^3\,\mu^6\,\alpha_2}{\big(\det Y_E\big)\,\big(\det Y_U\big)^{4/3}\,\big(\det Y_D\big)^{1/3}}\,\exp\Big(\frac{\pi}{\alpha_2}+\frac{5\pi}{3\alpha_1}\Big);$$

$$\text{RGI}_2=\frac{\boldsymbol{\mu}^3\,(\alpha_3)^3}{\mu^3\det Y_U\det Y_D}\,\exp\Big(\frac{2\pi}{\alpha_3}\Big),$$

respectively.

The scheme dependence of the above equations becomes essential starting from the order $O(\alpha^2,\alpha Y^2,Y^4)$ corresponding to the three-loop approximation for the $\beta$-functions and to the two-loop approximation for the anomalous dimensions.

# The three-loop check

The three-loop $\beta$-functions for the MSSM in the HD+MSL scheme have been calculated in

O. Haneychuk, V. Shirokova, K.S., JHEP **09** (2022), 189,

and in the $\overline{\text{DR}}$ scheme in

I. Jack, D. R. T. Jones, A. F. Kord, Annals Phys. **316** (2005), 213.

In the HD+MSL scheme the above equations are satisfied in the three-loop approximation independently of the regularization parameters, e.g.,

$$\Big[\Big(\frac{3}{\alpha_3}-\frac{2\pi}{\alpha_3^2}\Big)\beta_3-3+3\gamma_{\mu}-\gamma_{\text{det}\,Y_U}-\gamma_{\text{det}\,Y_D}\Big]_{\text{HD+MSL}}=O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big).$$

However, in the $\overline{\text{DR}}$ scheme these equations are not valid starting from the three-loop approximation, e.g.,

$$\Big[\Big(\frac{3}{\alpha_3}-\frac{2\pi}{\alpha_3^2}\Big)\beta_3-3+3\gamma_{\mu}-\gamma_{\text{det}\,Y_U}-\gamma_{\text{det}\,Y_D}\Big]_{\overline{\text{DR}}}=\frac{1}{2\pi^2}\Big(\frac{363\alpha_1^2}{400}+\frac{9\alpha_2^2}{16}-\frac{21\alpha_3^2}{8}\Big)$$
$$+\frac{1}{16\pi^3}\text{tr}(Y_U Y_U^+)\Big(\frac{13\alpha_1}{30}+\frac{3\alpha_2}{2}+\frac{8\alpha_3}{3}\Big)+\frac{1}{16\pi^3}\text{tr}(Y_D Y_D^+)\Big(\frac{7\alpha_1}{30}+\frac{3\alpha_2}{2}+\frac{8\alpha_3}{3}\Big)$$
$$-\frac{1}{(16\pi^2)^2}\Big[6\,\text{tr}\Big((Y_U Y_U^+)^2\Big)+6\,\text{tr}\Big((Y_D Y_D^+)^2\Big)+6\Big(\text{tr}(Y_U Y_U^+)\Big)^2+6\Big(\text{tr}(Y_D Y_D^+)\Big)^2$$
$$+2\,\text{tr}\Big(Y_E Y_E^+\Big)\,\text{tr}\Big(Y_D Y_D^+\Big)+4\,\text{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)\Big]+O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big).$$

# Conclusion

- The running of gauge and Yukawa couplings in various supersymmetric theories obeys some equations, which are exact in all orders of the perturbation theory.
- Due to these equations, various couplings in supersymmetric theories do not run independently.
- Some of the exact relations between running couplings in supersymmetric theories can be written in the form of RGIs, which, by definition, are scale independent in all orders.
- In particular, for $\mathcal{N}=1$ SQCD+SQED it is possible to construct an RGI from two gauge couplings.
- For MSSM there are two independent RGIs involving gauge and Yukawa couplings.
- The exact equations relating RGFs are valid only for certain renormalization prescriptions which include the HD+MSL scheme and, in general, do not include $\overline{\text{DR}}$ scheme.

Thank you for the attention!