On the several perturbation theory QCD representations for the e^+e^- annihilation to hadrons Adler function

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XXII Lomonosov Conference
on Elementary Particle Physics
August 25, 2025
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Intoductional coments

- 10th talk at Lomonosov Conferebce Series
- 4th e^+e^- annihilation Adler function related talk
  1997 year Lomonosov Conference ;
  1999 year Lomonosov Conference (Particle Physics at the start of New Millenium , pp. 43-52),
  2023 year Lomonosov Conference
- Returning to the topic in view of two talks at XXII Lomonsov Conference by
  BESIII collaboration related talk by G. S. Huang (21.08)
  KEDR collaboration related talk by K. Todyshev (23.08)
- And in view of XVII Academician Markov INR Conference (21 .05) with participation by
  I. Ya. Arefieva, D. I. Kazakov and V.A. Smirnov

2025 Markov Conference, INR

https://yandex.ru/video/preview/10823430920463103816

Problems to be touched

- High orders perturbative QCD and Renormalization Group related signals of the conformal symmetry violations ;
- Axial-Vector-Vector anomaly as the bridge between Deep Inelastic and Annihilation procecceses ;
- Bridges between DI sum rules and e^+e^--annihilation to hadrons D-function (and R-ratio) PT QCD expressions ;
- Perturbative QCD scale-scheme ambiguities ; consideration of theoretical uncertainties of PT QCD predictions

Processes to be considered

- The process e^+e^- - annihilation to hadrons. It is tested at different colliders. It is possible to consider e^+e^- annihilation into γ^* or Z^0, creating then hadrons. Novosibirsk, Bejing, KEK existing colliders and CEPC, CERN FCC etc.
- DIS processes give possibility to understand better the content of nucleon; JLAB, EIC (BNL)
- Is it possible to relate characteristics of definite annihilation (s-channel) and DIS (t-channel) processes ? The answer is - yes, through applying OPE to the AVV triangle diagram. What are the outcomes ?

Definitions of basic quantities

The e^+e^- to hadrons D function in QCD with $a_s = \alpha_s/\pi$

$$D(a_s(Q^2)) = Q^2 \int_0^\infty ds \frac{R^{th}_{e^+e^-}(a_s(s))}{(s+Q^2)^2} \to Q^2 \int_0^\infty ds \frac{R^{exp+th}_{e^+e^-}(s)}{(s+Q^2)^2},$$

$$R^{th}_{e^+e^-} = \sigma^{e^+e^- \to hadrons}_{tot}(a_s)/(\sigma_0(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha^2/(3s)).$$

$$\left(\frac{\partial}{\partial \ln \mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right) D(a_s) = 0, \frac{\partial a_s}{\partial \ln \mu^2} = \beta(a_s)$$

$$D(a_s) = (\mathbf{N_c = 3}) \left(\sum_i q_i^2\right) D^{NS}(a_s) + \left(\sum_i q_i\right)^2 D^{SI}(a_s)$$

While considering $\sigma_0(e^+e^- \to \gamma^* \to \mu^+\mu^-) = 4\pi\alpha^2/(3s)$ we fix RS procedure $\alpha = Z_3(\alpha)\alpha_B$ with $Z_3 = 1$. Then the expression for $R^{th}(s)$ is defined in the model $SU_c(3) + U(1)$ and not $SU_c(3) \times U(1)$ (in THEORY : γ^* (or decaying Z^0) is virtual

$\overline{\text{MS}}$-scheme results for $D^{NS}(a_s) = 1 + \sum_{n\geq 1} d_n a_s^n$

$$d_1 = \frac{3}{4}C_F; d_2 = -\frac{3}{32}C_F^2 - \left(\frac{11}{8} - \zeta_3\right)C_F T_f + \left(\frac{123}{32} - \frac{11}{4}\zeta_3\right)C_F C_A;$$

$$= 1.9857 - 0.1153 n_f; T_f = n_f/2$$

Chetyrkin, Kataev, Tkachov (79); numerical Dine, Sapirstein (1979); analytical Celmaster, Gonsalves (1980); unpublished Ross, Terrano, Wolfram (1978-1980-corrected by Ch,K,T)

$$d_3 = -\frac{69}{128}C_F^3 - \left(\frac{29}{64} - \frac{19}{4}\zeta_3 + 5\zeta_5\right)C_F^2 T_f + \left(\frac{151}{54} - \frac{19}{9}\zeta_3\right)C_F T_f$$
$$-\left(\frac{127}{64} + \frac{143}{16}\zeta_3 - \frac{55}{4}\zeta_5\right)C_F^2 C_A + \left(\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right)C_F C_A^2$$
$$-\left(\frac{485}{27} - \frac{112}{9}\zeta_3 - \frac{5}{6}\zeta_5\right)C_F C_A T_f = 18.243 - 4.216 n_f + 0.086 n_f^2$$

Gorishny,K,Larin (87-bug in SCHOONSCHIP programs); corrected and recalculated further in Gorishny,K,Larin (91); Surguladze, Samuel (91); Chetyrkin (97)

d_4 in the $\overline{\text{MS}}$-scheme

$$d_4 = \frac{d_F^{abcd} d_A^{abcd}}{d_R}\left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right) + n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}\left(-\frac{13}{16} - \zeta_3 - \frac{5}{2}\zeta_5\right)$$
$$+C_F^4\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right) + C_F^3 T_f\left(\frac{1001}{384} + \frac{99}{32}\zeta_3 - \frac{125}{4}\zeta_5 + \frac{105}{4}\zeta_7\right)$$
$$+C_F^2 T_f^2\left(\frac{5713}{1728} - \frac{581}{24}\zeta_3 + \frac{125}{6}\zeta_5 + 3\zeta_3^2\right)$$
$$+C_F T_f^3\left(-\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5\right)$$
$$+C_F^3 C_A\left(-\frac{253}{32} - \frac{139}{128}\zeta_3 + \frac{2255}{32}\zeta_5 - \frac{1155}{16}\zeta_7\right)$$
$$+C_F^2 T_f C_A\left(\frac{32357}{13824} + \frac{10661}{96}\zeta_3 - \frac{5155}{48}\zeta_5 - \frac{33}{4}\zeta_3^2 - \frac{105}{8}\zeta_7\right) +$$

d_4-continuation

$$+C_F C_A T_f^2\left(\frac{340843}{5184}-\frac{10453}{288}\zeta_3\right)$$
$$+C_F^2 C_A^2\left(-\frac{592141}{18432}-\frac{49325}{384}\zeta_3+\frac{6505}{48}\zeta_5+\frac{1155}{32}\zeta_7\right)$$
$$+C_F C_A^2 T_f\left(-\frac{4379861}{20736}+\frac{8609}{77}\zeta_3+\frac{18805}{288}\zeta_5-\frac{11}{2}\zeta_3^2+\frac{35}{16}\zeta_7\right)$$
$$+C_F C_A^3\left(\frac{52207039}{248832}-\frac{426223}{3456}\zeta_3-\frac{77995}{1152}\zeta_5+\frac{605}{32}\zeta_3^2-\frac{385}{64}\zeta_7\right)$$
$$=135.792-34.440n_f+1.875n_f^2-0.010n_f^3$$

Baikov, Chetyrkin, Kuhn (2008-2010); Confirmed by Herzog, Ruijl, Ueda, Vermaseren, Vogt (2017)

Perturbative QCD for LEP

Reports of the Working Group on Precision Calculations for the Z Resonace CERN 95-03 Yellow Report ; 410 pages
Eds. Dmitry Bardin; Wolfgang Hollik ; Gian Piero Passarino ;

- K.G. Chetyrkin, J. H. Kuhn, A. Kwiatkowski, QCD Corrections to the e^+e^- Cross-Section and the Z Boson Decay Rate , pp.175-263 ; Phys. Rept. 277 (1996) 189-281
- S.A.Larin, T. van Ritbergen, J. A. M. Vermaseren, The Large Quark Mass Expansion of $\Gamma(Z^0 \to hadrons)$ in Order α_s^3, pp. 265-274;
- J. Chyla, A. L. Kataev, Theoretical Ambiguities of QCD Predictions at the Z^0 Peak, pp. 313-340

S. G. Gorishnii, A.L. Kataev and S. A. Larin; The behaviour of $\sigma_{tot}(e^+e^- \to hadrons)$ in QCD and in theories with scalar quarks, Proceedings of "Physics of e^+e^- Interactions "; December 1987; Dubna E2-88-363 ; pp.33-60

Sum rules of lN and νN deep-inelastic scattering

$$S_{Bjp}(Q^2) = \int_0^1 dx[g_1^{(lp)}(x,Q^2) - g_1^{(ln)}(x,Q^2)] = \frac{1}{6}|\frac{g_A}{g_V}|C_{Bjp}(a_s(Q^2))$$

$$S_{GLS}(Q^2) = \frac{1}{2}\int_0^1 dx[F_3^{(\nu p)}(x,Q^2) + F_3^{(\nu n)}(x,Q^2)] = 3C_{GLS}(a_s)$$

$$C_{Bjp}(a_s) = C^{NS}(a_s) + C^{SI}_{Bjp}(a_s)$$

$$C_{GLS}(a_s) = C^{NS}(a_s) + C^{SI}_{GLS}(a_s)$$

Sum rules are used for extraction of α_s values and study of the contributions of high-twist non-perturbativec effects Non-singlet and singlet coefficient functions should be analysed separately

$\overline{\mathrm{MS}}$-scheme analytical results for $C^{NS}(a_s) = 1 + \sum_{n\geq 1} c_n a_s^n$

$$c_1 = -\frac{3}{4}C_F$$

$$c_2 = \frac{21}{32}C_F^2 + \frac{1}{2}C_F T_f - \frac{23}{16}C_F C_A = -4.583 + 0.333 n_f$$

Gorishny, Larin (1986)

$$c_3 = -\frac{3}{128}C_F^3 - \left(\frac{133}{576} + \frac{5}{12}\zeta_3\right)C_F^2 T_f - \frac{115}{216}C_F T_f^2$$
$$+\left(\frac{1241}{576} - \frac{11}{12}\right)C_F^2 C_A + \left(-\frac{5437}{864} + \frac{55}{24}\zeta_5\right)C_F C_A^2$$
$$+\left(\frac{3535}{864} + \frac{3}{4}\zeta_3 - \frac{5}{6}\zeta_5\right)C_F C_A T_f = -41.4399 + 7.6077 n_f - 0.1775 n_f^2$$

Larin,Vermaseren (1991)

c_4 coefficient in the $\overline{\mathrm{MS}}$-scheme

$$
\begin{aligned}
c_4 = \frac{d_F^{abcd} d_A^{abcd}}{d_R}\Big(-\frac{3}{16}+\frac{1}{4}\zeta_3+\frac{5}{4}\zeta_5\Big) + n_f\,\frac{d_F^{abcd} d_F^{abcd}}{d_R}\Big(\frac{13}{16}+\zeta_3-\frac{5}{2}\zeta_5\Big) \\
+C_F^4\Big(-\frac{4157}{2048}-\frac{3}{8}\zeta_3\Big)+C_F^3 T_f\Big(\frac{839}{2304}+\frac{451}{96}\zeta_3-\frac{145}{24}\zeta_5\Big) \\
+C_F^2 T_f^2\Big(-\frac{265}{576}+\frac{29}{24}\zeta_3\Big)+C_F T_f^3\Big(\frac{605}{972}\Big) \\
+C_F^3 C_A\Big(-\frac{3707}{4608}-\frac{971}{96}\zeta_3+\frac{1045}{48}\zeta_5\Big) \\
+C_F^2 T_f C_A\Big(-\frac{37403}{13824}-\frac{1289}{144}\zeta_3+\frac{275}{144}\zeta_5+\frac{105}{8}\zeta_7\Big)
\end{aligned}
$$

c_4-continuation

$$+C_F C_A T_f^2\left(-\frac{165283}{20736}-\frac{43}{144}\zeta_3+\frac{5}{12}\zeta_5-\frac{1}{6}\zeta_3^2\right)$$
$$+C_F^2 C_A^2\left(\frac{1071641}{55296}+\frac{1591}{144}\zeta_3-\frac{1375}{144}\zeta_5+\frac{385}{16}\zeta_7\right)$$
$$+C_F C_A^2 T_f\left(-\frac{1238827}{41472}+\frac{59}{64}\zeta_3-\frac{18855}{288}\zeta_5+\frac{11}{12}\zeta_3^2-\frac{35}{16}\zeta_7\right)$$
$$+C_F C_A^3\left(-\frac{8004277}{248832}+\frac{1069}{576}\zeta_3+\frac{12545}{1152}\zeta_5-\frac{121}{96}\zeta_3^2+\frac{385}{64}\zeta_7\right)$$

Baikov, Chetyrkin, Kuhn (2010) evaluated directly ; presented first in this forn by Kataev, Mikhailov (10-12) (memorial Theor Mat Fiz (2012) volume of Academician Tavkhelidze)

Special analytical structure of the CBK relation

Conformal symmetry based study Crewther (1972) and respecting QCD asymptotic freedom effects at $O(a_s^3)$ level Broadhurst, Kataev (1993) CBK relation . Property of the factorization of the QCD β-function is outlined. Confirmed at the $O(a_s^4)$-level by Chetyrkin, Baikov, Kuhn (2010)

$$C_{NS}(a_s)D^{NS}(a_s) = 1 + ZERO(C_F^n a_s^n) + ZERO(C_F^k C_A^m a_s^{k+m}) +$$

$$+\frac{\beta^{(3)}(a_s)}{a_s}\Big[S_1 C_F a_s + \Big(S_2 T_f N_f + S_3 C_A + S_4 C_F\Big) C_F a_s^2$$

$$+a_s^3\Big(S_5 C_F^3 + S_6 C_F^2 T_f + S_7 C_F T_f^2 + S_8 C_F C_A^2 + S_9 C_F C_A T_f + S_{10} C_F C_A^2$$

$$S_1 = -\frac{21}{8} + 3\zeta_3, S_2 = \frac{163}{24} - \frac{19}{3}\zeta_3$$

$$S_3 = -\frac{629}{32} + \frac{221}{12}\zeta_3 \ , S_4 = \frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5, S_5 - S_{10} - analytical$$

In view of conformal symmetry applied to AVV scale-independent coefficients in C_{NS} and D^{NS} are cancelling

Factorization property of the QCD β-function

This property is non-trivial even in QED when $C_A = 0$. In general β-function is responsible for effects of conformal symmetry violation. Expressions in MS-like schemes

$$\beta(a_s) = -\sum_{n\geq 0} \beta_n a_s^n \ , \ \beta_0 = (\frac{11}{3}C_A - \frac{4}{3}Tf)\frac{1}{4}$$

$$\beta_1 = (\frac{34}{3}C_A^2 - \frac{20}{3}C_A T_f - 4C_F T_f)\frac{1}{16} \ ,$$

$$\beta_2 = (\frac{2857}{54}C_A^3 + 2C_F^2 T_f - \frac{205}{9}C_F C_A T_f - \frac{1415}{27}C_A^2 T_f + \frac{44}{9}C_F T_f^2 + \frac{158}{27}C_A T_f^2\Big)\frac{1}{64}$$

O.V. Tarasov,A.A. Vladimirov, A.A. Zharkov (1980); S.A.Larin, J.A.M. Vermaseren (1993); Available 4 and 5 loop QCD β-function terms will be not considered.

Massless $\overline{\text{MS}}$ Broadhurst-Kataev (93) renormalon related exp

$$\sum_{n<10} R_n x^n = 3x + \left[-22 + 16\zeta_3\right]x^2 + \left[\tfrac{4832}{27} - \tfrac{1216}{9}\zeta_3\right]x^3 + \left[-\tfrac{392384}{243} + \tfrac{25984}{27}\zeta_3 + \tfrac{1280}{3}\zeta_5\right]x^4$$
$$+ \left[\tfrac{11758720}{729} - \tfrac{5073920}{729}\zeta_3 - \tfrac{194560}{27}\zeta_5\right]x^5 + \left[-\tfrac{3499697920}{19683} + \tfrac{357201920}{6561}\zeta_3 + \tfrac{20787200}{243}\zeta_5 + \tfrac{71680}{3}\zeta_7\right]x^6$$
$$+ \left[\tfrac{381559797760}{177147} - \tfrac{9308446720}{19683}\zeta_3 - \tfrac{2029568000}{2187}\zeta_5 - \tfrac{5447680}{9}\zeta_7\right]x^7$$
$$+ \left[-\tfrac{5056220794880}{177147} + \tfrac{2445582254080}{531441}\zeta_3 + \tfrac{200033075200}{19683}\zeta_5 + \tfrac{814858240}{81}\zeta_7 + \tfrac{194969600}{81}\zeta_9\right]x^8$$
$$+ \left[\tfrac{5908327309475840}{14348907} - \tfrac{239732713062400}{4782969}\zeta_3 - \tfrac{20850920652800}{177147}\zeta_5 - \tfrac{318236262400}{2187}\zeta_7 - \tfrac{59270758400}{729}\zeta_9\right]x^9$$

$$\sum_{n<10} K_n x^n = -3x + 8x^2 - \tfrac{920}{27}x^3 + \tfrac{38720}{243}x^4 - \tfrac{238976}{243}x^5 + \tfrac{130862080}{19683}x^6 - \tfrac{10038092800}{177147}x^7$$
$$+ \tfrac{274593587200}{531441}x^8 - \tfrac{82519099473920}{14348907}x^9$$

$$\sum_{n<10} S_n x^n = \left[-\tfrac{21}{2} + 12\zeta_3\right]x + \left[\tfrac{326}{3} - \tfrac{304}{3}\zeta_3\right]x^2 + \left[-\tfrac{9824}{9} + \tfrac{6496}{9}\zeta_3 + 320\zeta_5\right]x^3$$
$$+ \left[\tfrac{2760448}{243} - \tfrac{1268480}{243}\zeta_3 - \tfrac{48640}{9}\zeta_5\right]x^4 + \left[-\tfrac{280736320}{2187} + \tfrac{89300480}{2187}\zeta_3 + \tfrac{5196800}{81}\zeta_5 + 17920\zeta_7\right]x^5$$
$$+ \left[\tfrac{10320047360}{6561} - \tfrac{2327111680}{6561}\zeta_3 - \tfrac{507392000}{729}\zeta_5 - \tfrac{1361920}{3}\zeta_7\right]x^6$$
$$+ \left[-\tfrac{3723517199360}{177147} + \tfrac{611395563520}{177147}\zeta_3 + \tfrac{50008268800}{6561}\zeta_5 + \tfrac{203714560}{27}\zeta_7 + \tfrac{48742400}{27}\zeta_9\right]x^7$$
$$+ \left[\tfrac{485484017500160}{1594323} - \tfrac{59933178265600}{1594323}\zeta_3 - \tfrac{5212730163200}{59049}\zeta_5 - \tfrac{79559065600}{729}\zeta_7 - \tfrac{14817689600}{243}\zeta_9\right]x^8$$
$$+ \Big[-\tfrac{7616109282344960}{1594323} + \tfrac{726735764193280}{1594323}\zeta_3 + \tfrac{195646580326400}{177147}\zeta_5 + \tfrac{1120185221120}{729}\zeta_7$$
$$+ \tfrac{316630630400}{243}\zeta_9 + \tfrac{7821721600}{27}\zeta_{11}\Big]x^9$$

Few words on factorization of the β-function

- Is important feature. Measure of violation of the CS in renormalized QFT models. Step to proof in all orders Crewther (96).
- Step to independent confirmation V.Braun, Korchemsky, Muller (03)
- Factorization is valid in gauge-independent schemes Garkusha, AK,Molokoedov (18) and gauge-dependent schemes Gracey, Mason (23)
- Considerations of static potential vs cusp anomalous dimension= Wilson loops related Grozin,Henn, Korchemsky , Merquard (16); Grozin (23)

In t 'Hooft scheme no property of factorization, though effects of β_0 and β_1 are seen Garkusha, AK (11). **In the diagrammatic $\overline{MS}$-relatesd schemes** CBK relation can be re-written as

$$C_{NS}(a_s)D^{NS}(a_s) = 1 + \sum_{n\geq 0}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

$P_0(a_s) = 0$ Effect of conformal symmetry. AK,Mikhailov (10-12)

The $\{\beta\}$-expansion for the RG-invariant quantities

$$D^{ns}(a_s) = 1 + d_1 a_s + d_2(n_f)a_s^2 + d_3(n_f)a_s^3 + d_4(n_f)a_s^4 + O(a_s^5)$$

In the MS-like schemes β-expansion prescription is:

$$d_1 = d_1[0]$$

$$d_2(n_f) = \beta_0 d_2[1] + \mathbf{d_2[0]}\textbf{Brodsky,Lepage, Mackenzie BLM (1983}$$

$$d_3(n_f) = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + \mathbf{d_3[0]},$$

$$d_4(n_f) = \beta_0^3 d_4[3] + \beta_2 d_4[0,0,1] + \beta_1\beta_0 d_4[1,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1]$$
$$+\beta_0 d_4[1] + \mathbf{d_4[0]}; \ldots$$

Suggested by **Mikhailov (Quarks2004, hep-ph.0411397 ; JHEP(07))** Further on Kataev, Mikhalov (12,15,16) ; Brodsky,Wu, Mojaza et al(12-25) ; Cvetic,Kataev(16) ; Kataev,Molokoedov (22,23) ; Baikov, Mikhailov (22-23) ; Mikhailov (24) **diagrammatic but or (!) extanded QCD and in part SQCD related study**

The $\{\beta\}$ expanded QCD terms for D^{ns} in $SU(N_c)$

Using the Cvetic,Kataev (16) $\overline{MS}$-scheme factorized model , motivated by Kataev,Mikhailov (12) CBK-related consideration

$$D^{ns}(a_s) = 1 + \sum_{n\geq 0} \left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

we obtain the results, which differ in part from obtained in **QCD+gluino theory Mikhailov (07), Kataev,Mikhailov (14;15;16) diagrammatic** analysis

$$d_1[0] = \frac{3}{4}C_F \;\; d_2[0] = \left(-\frac{3}{32}C_F^2 + \frac{1}{16}C_F C_A\right) \;\; d_2[1] = \left(\frac{33}{8} - 3\zeta_3\right)C_F$$

$$d_3[0] = -\frac{69}{128}C_F^3 - \left(\frac{101}{256} - \frac{33}{16}\zeta_3\right) \neq +\frac{\mathbf{71}}{\mathbf{64}} \;\; C_F^2 C_A$$
$$-\left(\frac{53}{192} + \frac{33}{16}\zeta_3\right) \neq +\left(\frac{\mathbf{523}}{\mathbf{768}} - \frac{\mathbf{27}}{\mathbf{8}}\boldsymbol{\zeta_3}\right) \;\; C_F C_A^2$$

As the result one has $d_3[0] = -23.227 \neq -\mathbf{35.87}$,

The $\{\beta\}$ expanded QCD expression for d_4 for Adler e^+e^--function

$$d_4[0] = \left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} - \left(\frac{13}{16} + \zeta_3 - \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f$$

$$+\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right)C_F^4$$

$$-\left(\frac{3509}{1536} + \frac{73}{128}\zeta_3 + \frac{165}{32}\zeta_5\right) \neq -\left(\frac{\mathbf{2409}}{\mathbf{512}} + \frac{\mathbf{27}}{\mathbf{16}}\zeta_{\mathbf{3}}\right) \quad C_F^3 C_A$$

$$+\left(\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5\right) \neq -\left(\frac{\mathbf{3105}}{\mathbf{1024}} + \frac{\mathbf{81}}{\mathbf{32}}\zeta_{\mathbf{3}}\right) \quad C_F^2 C_A^2$$

$$\left(-\frac{30863}{36864} - \frac{147}{128}\zeta_3 + \frac{165}{64}\zeta_5\right) \neq \left(\frac{\mathbf{68047}}{\mathbf{12288}} + \frac{\mathbf{8113}}{\mathbf{512}}\zeta_{\mathbf{3}} - \frac{\mathbf{3555}}{\mathbf{128}}\zeta_{\mathbf{5}}\right)C_F C_A^3$$

$d_4[0] = +81.157 \neq -\mathbf{98}$ Differ from **diagrammatic related expression of Mikhailov (22-24), obtained using Chetyrkin (22) and Zoller (16) eQCD calculations** supported by diagrammatic QCD Ball, Beneke, V. Braun (95) .

D-function at the $O(a_s^4)$-level and PMC/BLM - representations

Compare $\overline{\text{MS}}$-scheme D-function with the **scale independent** coefficients Cvetic,K (16) -model with with Mikhailov (07;..;24) diagrammatic **though** quantity dependent Extended QCD

$$D_{\overline{\text{MS}}}(a_s) = 3\sum_f Q_f^2\Big(1 + a_s + (1.9857 - 0.1153n_f)a_s^2$$

$$+(18.243 - 4.216n_f + 0.086n_f^2)a_s^3$$

$$+(135.792 - 34.440n_f + 1.875n_f^2 - 0.010n_f^3)a_s^4\Big)$$

$$D^{CK}_{PMC/BLM}(a_s) = 3\sum_f Q_f^2\Big(1 + a_* + \frac{1}{12}a_*^2 - 23.227a_*^3$$

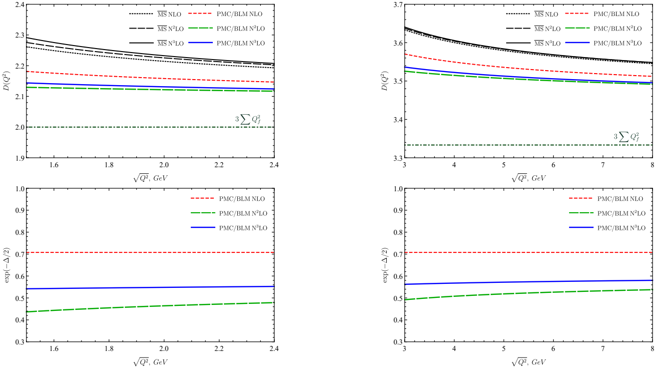
$$+(81.157 - 0.0080n_f)a_*^4\Big)$$

$$D^{M}_{PMC/BLM}(a_s) = 3\sum_f Q_f^2\Big(1 + a_{**} + \frac{1}{12}a_{**}^2 - 35.87a_{**}^3$$

$$+(-98 - 0.0080n_f)a_{**}^4\Big)$$

Basis of PMC/BLM proposed in **Brodsky, Wu (12-25) idea**

PMC/BLM vs massless $\overline{\text{MS}}$: AK,Molokoedov PRD 108 (23)



Experimental related data are higher then $\overline{MS}$ Eidelman, Jegerlehner, AK, Veretin (98); Davier et al (23).

PMC as generalization of BLM X. G. Wu, J. M. Shen, B. L. Du, X. D. Huang, S. Q. Wang and S. J. Brodsky, Prog. Part. Nucl. Phys. **108** (2019), 103706

Conclusion from K,Molokoedov PRD 108 (2023) 096027

Apologizes to our guests from China, but since is science
As seen from Fig. 1a, the application of the PMC-BLM procedure to the massless MS Adler function PT approximations is leading to moving the considered curves lower away fromthe experimentally based results for the Adler function in the considered kinematical region. Therefore, better not to use the PMC-BLM in the process of comparison with the existing experimental data and, in particular, the ones provided by the e^+e^- *colliders. PMC/BLM are qualitatively closer to "Finite QED" . May lead to UNDERESTIMATE of theory QCD ambiguities . .*
Let us wait also A. Arbuzov and U. Voznaya today e^+e^- **related talks**

Conclusion

The BESIII and KEDR e^+e^- hadrons R(s) data may be used for more detailed considerations of Adler-function extraction from S. Eidelman and F. Jegerlehner (1995) related bank of e^+e^- data

For the latest analysis see

M. Davier, D. Díaz-Calderón, B. Malaescu, A. Pich, A. Rodríguez-Sánchez and Z. Zhang, "The Euclidean Adler function and its interplay with $\Delta\alpha_{\rm QED}^{\rm had}$ and α_s," JHEP **04** (2023), 067 [arXiv:2302.01359 [hep-ph]]. and 1999 year Phys Lett Eidelman, Jegerlehner, AK and Veretin