Quantum field theory description of neutrino oscillations in external fields

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Outline of talk

- Quantum mechanical (QM) description of neutrino oscillations
  - ✓ Oscillations in vacuum
  - ✓ Effective Hamiltonian
- Neutrino interaction with external fields
  - ✓ Effective Hamiltonian treatment of oscillations in matter and a magnetic field
- Shortcomings of the QM description
- Model of neutrino oscillations in space
- Quantum field theory (QFT) treatment of neutrino oscillations: neutrinos as virtual particles
  - ✓ Calculation of the matrix element
  - ✓ Dyson equation for the dressed propagators and its solution
  - ✓ Flavor oscillation in matter
  - ✓ Spin-flavor precession in a magnetic field
- Conclusion

References

- M.Dvornikov, Quantum field theory treatment of neutrino spin-flavor oscillations in a magnetic field, arxiv:2504.14726
- M.Dvornikov, Quantum field theory treatment of neutrino flavor oscillations in matter, Phys. Rev. D 111, 056009 (2025); arxiv:2411.19120

Neutrino oscillations in a nutshell

- We have three types (flavors) of active neutrinos: $(\nu_e, \nu_\mu, \nu_\tau)$
- These flavor neutrinos interact with other fermions in the standard model
- Flavor neutrinos dot not have definite masses. They are the superposition of neutrino mass states:

$$\nu_\lambda = \sum_a U_{\lambda a} \psi_a \qquad (U_{\lambda a}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\lambda = e, \mu \quad a = 1, 2$$

Mixing matrix

- While evolving in space, one neutrino flavor is converted into another. This process is called neutrino flavor oscillations

QM description of neutrino oscillations

Historically, flavor oscillations have been considered within the framework of QM. The probabilities to detect certain neutrino flavors are consistent with experimental data in almost all situations

Suppose that one has only an electron neutrino initially $\nu_e(0) = 1, \quad \nu_\mu(0) = 0$

Mass states evolve as plane waves

$$\psi_a(x,t) \propto e^{-iE_a t + ipx} \qquad E_a = \sqrt{p^2 + m_a^2} \approx p + \frac{m_a^2}{2p}$$

One gets the evolution of flavor states and the transition probability

$$\nu_\lambda(t) = \sum_{a,\lambda'} U_{\lambda a} U^*_{\lambda' a} \exp\left(-i\frac{m_a^2}{2p}t\right)\nu_{\lambda'}(0) \qquad P_{\nu_e \to \nu_\mu}(t) = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}t\right)$$

Effective Hamiltonian

Mass states obey the effective Schrodinger equation

$$i\dot{\nu}_m = H_m \nu_m \qquad H_m = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \approx \frac{1}{4E}\begin{pmatrix} -\Delta m^2 & 0 \\ 0 & \Delta m^2 \end{pmatrix}, \quad \Delta m^2 = m_2^2 - m_1^2 > 0$$

Schrodinger equation can be written down for flavor states

$$i\dot{\nu}_f = H_f \nu_f \qquad H_f = U H_m U^\dagger = \frac{\Delta m^2}{4E}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

Initial condition $\nu_f(0) = \nu_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution of the Schrodinger equation

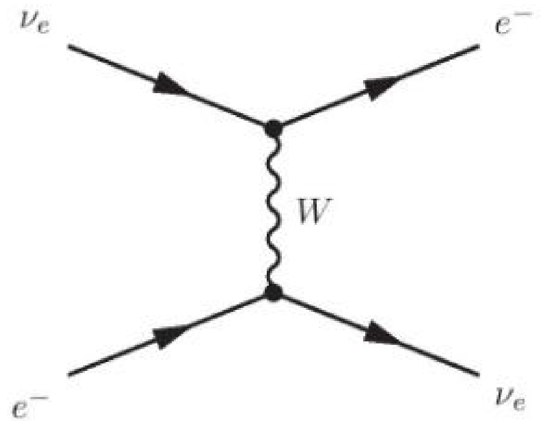
$$\nu_f(t) = \begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix} = e^{-iH_f t}\nu_f(0) = \begin{pmatrix} \cos\left(\frac{\Delta m^2}{4E}t\right) + i\sin\left(\frac{\Delta m^2}{4E}t\right)\cos 2\theta \\ -i\sin\left(\frac{\Delta m^2}{4E}t\right)\sin 2\theta \end{pmatrix}$$

Transition probability

$$P_{\nu_e\to\nu_\mu}(t) = |\nu_\mu(t)|^2 = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}t\right)$$

Neutrino interaction with background matter

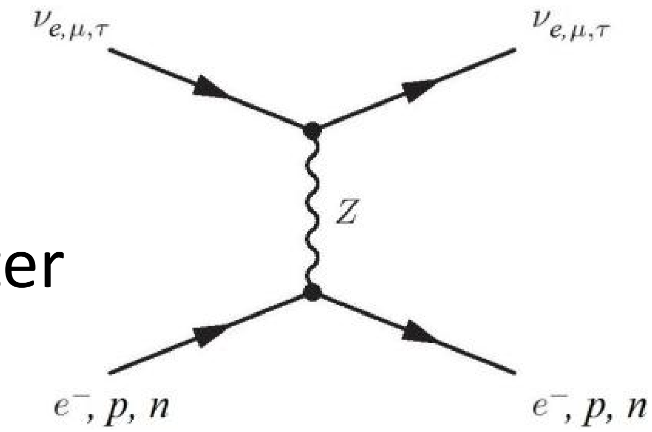
Lagrangian of the effective interaction



$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}}\bar{\nu}\gamma_\mu^L\nu \cdot \bar{f}(C_V\gamma^\mu - C_A\gamma^\mu\gamma^5)f$$

Forward scattering approximation

$$\langle \bar{f}\gamma^\mu f\rangle = j_f^\mu, \quad \langle \bar{f}\gamma^\mu\gamma^5 f\rangle = \lambda_f^\mu$$



$$\mathcal{L} = -\frac{G_F}{\sqrt{2}}\bar{\nu}\gamma_\mu(1-\gamma^5)\nu \cdot V^\mu \quad V^\mu = \sum_f [j_f^\mu q_f^{(1)} + \lambda_f^\mu q_f^{(2)}]$$

The effective potentials in nonmoving and unpolarized matter

$$V_{\nu_e} = \sqrt{2}G_{\mathrm{F}}\left(n_e - \frac{1}{2}n_n\right), \quad V_{\nu_\mu} = -\frac{G_{\mathrm{F}}}{\sqrt{2}}n_n$$

Neutrino oscillations in matter

Effective Hamiltonian

$$H_f = \begin{pmatrix} -\frac{\Delta m^2}{4E}\cos 2\theta + \sqrt{2}G_{\mathrm{F}}n_e & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta \end{pmatrix}$$

Transition probability

$$P_{\nu_e\to\nu_\mu}(t) = \frac{\left(\frac{\Delta m^2}{4E}\sin 2\theta\right)^2}{\left(\frac{\Delta m^2}{4E}\sin 2\theta\right)^2 + \left(\frac{\Delta m^2}{4E}\cos 2\theta - \frac{G_{\mathrm{F}}n_e}{\sqrt{2}}\right)^2}\sin^2\left(\sqrt{\left(\frac{\Delta m^2}{4E}\sin 2\theta\right)^2 + \left(\frac{\Delta m^2}{4E}\cos 2\theta - \frac{G_{\mathrm{F}}n_e}{\sqrt{2}}\right)^2}\,t\right)$$

Assuming that the matter density slowly changes along the neutrino trajectory, one gets that the transition probability increases if

$$\frac{\Delta m^2}{4E}\cos 2\theta = \frac{G_{\mathrm{F}}n_e}{\sqrt{2}}$$

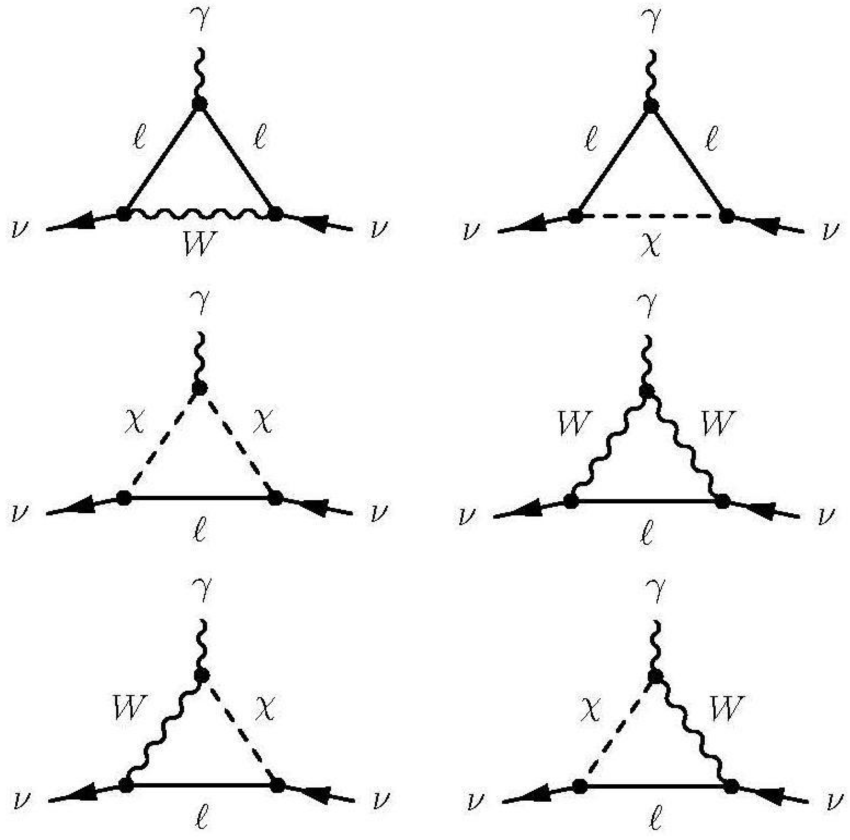
It is called the Mikheyev-Smirnov-Wolfenstein effect

Neutrino electrodynamics

- Neutrinos are electrically neutral particles
- Nevertheless, they can have anomalous magnetic moments
- Lagrangian of the electromagnetic interaction

$$\mathcal{L} = -\frac{\mu_{ab}}{2}\bar{\psi}_a \sigma_{\mu\nu} \psi_b F^{\mu\nu} \quad (M_{\lambda\lambda'}) = U(\mu_{ab})U^\dagger$$

- Dirac neutrinos can have both diagonal and transition magnetic moments
- Majorana neutrinos han have only transition moments. Matrix of magnetic moments is hermitian and purely imaginary
- Giunti et al. (2024) recently reviewed neutrino electromagnetic properties
- In presence of a magnetic field, a neutrino beam can change both polarization and flavor content. It is called the neutrino spin-flavor precession



Spin-flavor precession of Majorana neutrinos

- We consider two Majorana neutrinos in the basis $(\nu_e, \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu)$
- Magnetic field is transverse
- Effective Hamiltonian

$$H_f = \begin{pmatrix} -\frac{\Delta m^2}{4E_\nu}\cos 2\theta & \frac{\Delta m^2}{4E_\nu}\sin 2\theta & 0 & \mathrm{i}\mu B \\ \frac{\Delta m^2}{4E_\nu}\sin 2\theta & \frac{\Delta m^2}{4E_\nu}\cos 2\theta & -\mathrm{i}\mu B & 0 \\ 0 & \mathrm{i}\mu B & -\frac{\Delta m^2}{4E_\nu}\cos 2\theta & \frac{\Delta m^2}{4E_\nu}\sin 2\theta \\ -\mathrm{i}\mu B & 0 & \frac{\Delta m^2}{4E_\nu}\sin 2\theta & \frac{\Delta m^2}{4E_\nu}\cos 2\theta \end{pmatrix}$$

Exact solution satisfying $\nu(0) = \nu_e$

$$\nu(t) = \frac{1}{2\mathfrak{E}_\nu} \begin{pmatrix} \frac{(\mu B)^2 + \left(\frac{\Delta m^2}{4E_\nu}\right)^2 \sin^2 2\theta}{\mathfrak{E}_\nu + \frac{\Delta m^2}{4E_\nu}\cos 2\theta} e^{-\mathrm{i}\mathfrak{E}_\nu t} + \left(\mathfrak{E}_\nu + \frac{\Delta m^2}{4E_\nu}\cos 2\theta\right) e^{\mathrm{i}\mathfrak{E}_\nu t} \\ -\mathrm{i}\frac{\Delta m^2}{2E_\nu}\sin 2\theta \sin \mathfrak{E}_\nu t \\ 0 \\ -2\mu B \sin \mathfrak{E}_\nu t \end{pmatrix}$$

- Transition probability for the channel $\nu_e \to \bar{\nu}_\mu$

$$\mathfrak{E}_\nu = \sqrt{\left(\frac{\Delta m^2}{4E_\nu}\right)^2 + (\mu B)^2}$$

$$P_{\nu_e \to \bar{\nu}_\mu}(t) = |\bar{\nu}_\mu(t)|^2 = \frac{(\mu B)^2}{(\mu B)^2 + \left(\frac{\Delta m^2}{4E_\nu}\right)^2} \sin^2\left(\sqrt{(\mu B)^2 + \left(\frac{\Delta m^2}{4E_\nu}\right)^2}\, t\right)$$

Shortcomings of the QM approach

- While analyzing carefully the QM approach, one reveals the following shortcomings:
  - do different neutrino types have equal momenta, or velocities, or energies?
  - do neutrino oscillations happen in space or in time?
  - how spinor structure of the neutrino wavefunction affects neutrino oscillations?

  etc. (see, e.g., review by Naumov & Naumov (2020)).
- One has to develop the formalism for neutrino flavor oscillations based on QFT.

Model of neutrino flavor oscillations

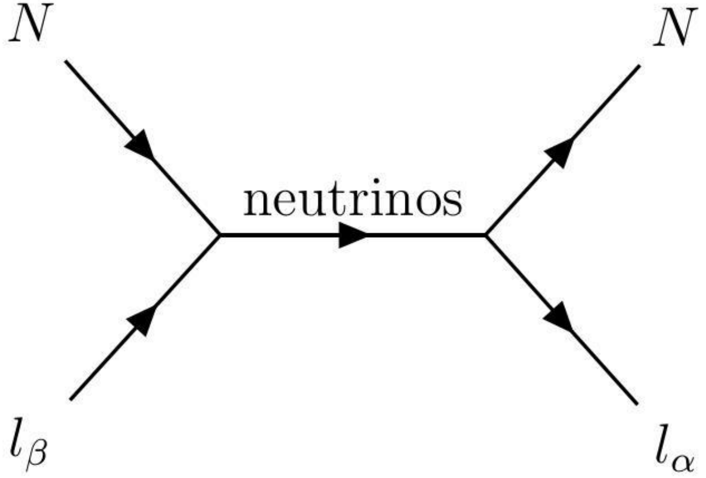
- We do not detect neutrinos in an experiment.
- One observes charged leptons which a neutrino interacts with.
- If the flavor content of leptons in a detector differs from that in a source, one can speak about neutrino oscillations.
- Charged leptons have definite masses. Thus, they in- and out-states are well defined for them.
- We assume that a source and a detector is made of heavy nuclei which charged leptons interact with.
- Neutrinos are virtual particles in this approach.
- Originally, this neutrino oscillations model was proposed by Kobzarev et. al. (1982); Giunti et al. (1993); Grimus & Stockinger (1996).

QFT applications for neutrino oscillations

- Giunti et al. (1993), Naumov & Naumov (2010): Consideration of wave packets of charged leptons allows one to consider decoherence effects in oscillations
- Egorov & Volobuev (2018): Modified propagators as the alternative for the wave packets formalism
- Cardall & Chung (1999); Akhmedov & Wilhelm (2013): QFT application for neutrino oscillations in matter
- Egorov & Volobuev (2019): QFT description of the spin-flavor precession in case of diagonal magnetic interaction
- Beuthe (2003); Naumov & Naumov (2020): Reviews on QFT in neutrino oscillations

QFT description in frames of the model

We consider the following Feynman diagram



$$S = -\frac{1}{2}\left(\sqrt{2}G_{int}\right)^2 \int \mathrm{d}^4x\mathrm{d}^4y \left\langle l_\alpha \middle| T\left\{ j_\mu^\dagger(x) J^\mu(x) j^\nu(y) J_\nu^\dagger(y) \right\} \middle| l_\beta \right\rangle$$ The S- matrix element

$$\mathcal{L}_{\text{int}} = \sqrt{2}G_{int} j_\mu J^\mu$$

Lagrangian of the lepton and nuclear currents interaction

$$j_\mu = \sum_\lambda \bar{\nu}_{\lambda \mathrm{L}} \gamma_\mu l_{\lambda \mathrm{L}}$$ Lepton current

$$J_\mu(x) \propto \delta_{\mu 0}\delta(\mathbf{x} - \mathbf{x}_2)$$

$$J_\nu(y) \propto \delta_{\nu 0}\delta(\mathbf{y} - \mathbf{x}_1)$$

Nuclei are supposed to be heavy. Thus they do not change their positions in a source ($\mathbf{x}_1$) and in a detector ($\mathbf{x}_2$)

Matrix element describing neutrino oscillations

We assume that neutrino mass eigenstates are virtual particles

$$\nu_\lambda = \sum_a U_{\lambda a} \psi_a$$

One gets for the S-matrix in the approximation of the plane waves leptons wavefunctions

$$S = -\frac{G_{int}^2 e^{-\mathrm{i}\mathbf{p}_\alpha \mathbf{x}_2 + \mathrm{i}\mathbf{p}_\beta \mathbf{x}_1}}{2V\sqrt{E_\alpha E_\beta}} 2\pi\delta(E_\alpha - E_\beta)\mathcal{M}_{\beta\to\alpha}$$

Matrix element $$\mathcal{M}_{\beta\to\alpha} = \sum_{ab} U_{\alpha a} U_{\beta b}^* \boldsymbol{\kappa}_-^\dagger(p_\alpha) \left(\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \Sigma_{ab}(E, \mathbf{q}) e^{\mathrm{i}\mathbf{q}\mathbf{L}} \right) \boldsymbol{\kappa}_-(p_\beta)$$

$$E = (E_\alpha + E_\beta)/2$$

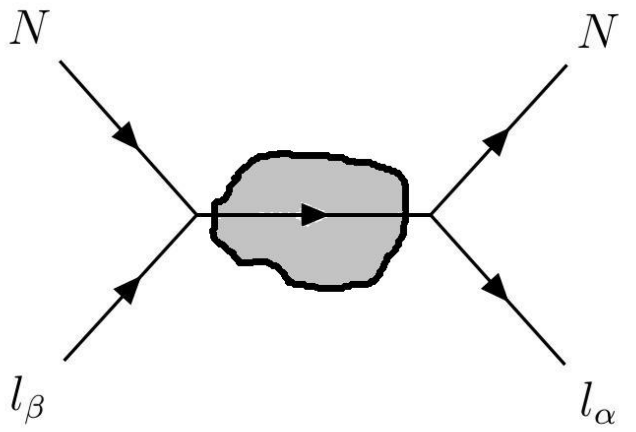
$$\left\langle 0 \middle| T\left\{ \eta_a(x) \eta_b^\dagger(y) \right\} \middle| 0 \right\rangle = \Sigma_{ab}(x-y)$$ Neutrino mass eigenstates propagators

In vacuum $\Sigma_{ab} \propto \delta_{ab}$

In this case, one can reproduce the expressions for the probabilities of neutrino oscillations in vacuum $P_{\beta\to\alpha} \propto \left|\mathcal{M}_{\beta\to\alpha}\right|^2$ (Kobzarev et al., 1982; Grimus & Stockinger, 1996).

Matrix element in background matter

- We consider the situation when the space between a source and a detector is filled with matter which a neutrino interacts electroweakly with.
- Our goal is to find the propagators Σ_{ab} in this case
- Matrix element and the probabilities are obtained after the computation of the 3D Fourier transform from the dressed and averaging of the results over the leptons states.



Wave equations for neutrinos in background matter

$$\mathcal{L} = \sum_{\lambda\lambda'} \bar{\nu}_\lambda \left[\delta_{\lambda\lambda'} \mathrm{i}\gamma^\mu \partial_\mu - m_{\lambda\lambda'} - V_{\lambda\lambda'} \frac{1}{2} \gamma^0 (1-\gamma^5) \right] \nu_{\lambda'}$$

$$V_e = \sqrt{2} G_\mathrm{F} \left(n_e - \frac{1}{2} n_n \right) \qquad V_\mu = -\frac{G_\mathrm{F}}{\sqrt{2}} n_n$$

Lagrangian for flavor neutrinos in matter. Neutrino interaction with matter is diagonal in neutrino types $(V_{\lambda\lambda'}) = diag(V_e, V_\mu)$

Mass matrix is diagonal in the mass basis. However, the matter interaction is not

$$\mathcal{L} = \sum_{ab} \bar{\psi}_a \left[\delta_{ab} \left(\mathrm{i}\gamma^\mu \partial_\mu - m_a \right) - g_{ab} \frac{1}{2} \gamma^0 (1-\gamma^5) \right] \psi_b$$

It results to the fact the coupling between the Dirac equations for massive neutrinos

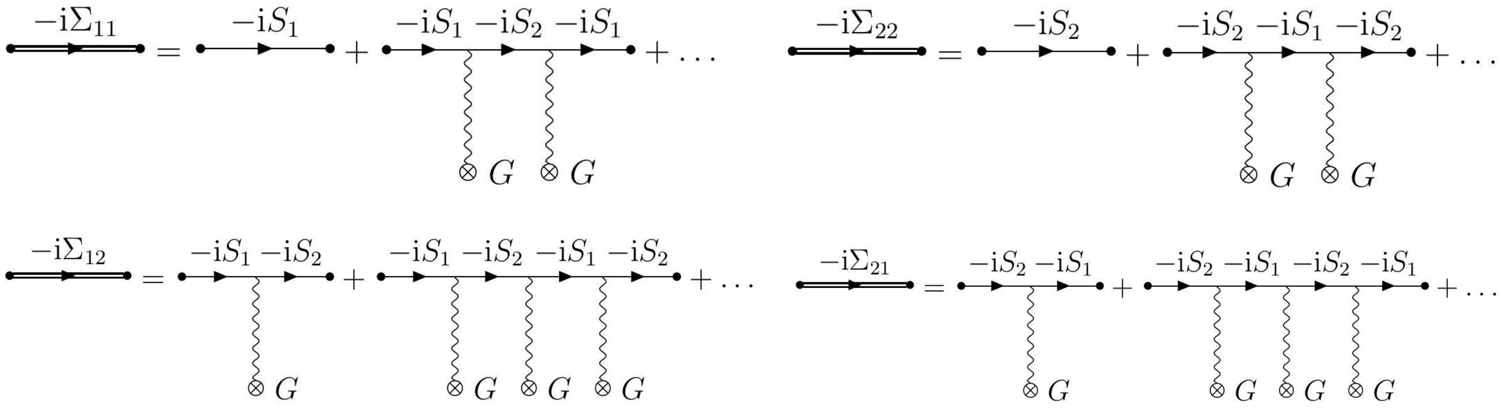
$$\left[\mathrm{i}\gamma^\mu \partial_\mu - m_1 - \frac{g_1}{2} \gamma^0 (1-\gamma^5) \right] \psi_1 = \frac{g}{2} \gamma^0 (1-\gamma^5) \psi_2, \quad \left[\mathrm{i}\gamma^\mu \partial_\mu - m_2 - \frac{g_2}{2} \gamma^0 (1-\gamma^5) \right] \psi_2 = \frac{g}{2} \gamma^0 (1-\gamma^5) \psi_1$$

Propagators of massive neutrinos in matter

One has to solve the 8x8 system of equations to find the propagators for two neutrinos

$$\begin{pmatrix} \mathrm{i}\gamma^\mu\partial_\mu - m_1 - G_1 & -G \\ -G & \mathrm{i}\gamma^\mu\partial_\mu - m_2 - G_2 \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{21} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix} = 1_{8\times 8}\,\delta(x)$$

One can look for the exact propagators in frames of the perturbation theory and sum all the terms in the perturbative series



The result of the summation of all the terms in Feynman diagrams is represented as the Dyson equations

$$(-\mathrm{i}S_1)^{-1} = (-\mathrm{i}\Sigma_{11})^{-1} + G(-\mathrm{i}S_2)G, \quad (-\mathrm{i}S_2)^{-1} = (-\mathrm{i}\Sigma_{22})^{-1} + G(-\mathrm{i}S_1)G$$

$$[(-\mathrm{i}S_1)G(-\mathrm{i}S_2)]^{-1} = (-\mathrm{i}\Sigma_{12})^{-1} + G, \quad [(-\mathrm{i}S_2)G(-\mathrm{i}S_1)]^{-1} = (-\mathrm{i}\Sigma_{21})^{-1} + G$$

$S_{1,2}$ are the diagonal propagators whichalso account for the diagonal interaction with matter

$$(\mathrm{i}\gamma^\mu\partial_\mu - m_{1,2} - G_{1,2})S_{1,2} = \delta(x)$$

Majorana neutrinos

- While solving the Dyson equation for Σ_{12} one has to deal with the inverse operator G^{-1}
- For Dirac neutrinos $G = \gamma^0(1-\gamma^5)/2$
- It is the projection operator which does not have a reciprocal G^{-1}
- However, for Majorana neutrinos G = g и $G_{1,2} = g_{1,2}$ are the numbers. Thus, Dyson equations for the dressed propagators are meaningful
- Modern models for the mass generation, predicting small masses of active neutrinos, involve Majorana mass states
- In priciple, one Dirac neutrino is equivalent to two Majorana particles with equal masses, $m_1=m_2$ and $m_3=m_4$. However, in this case, Feynman diagrams leading to the Dyson equations for the dressed propagators are branching, since one has to account for the transitions 1⟶3, 1⟶4, 2⟶3, 2⟶4 etc. Summing of the corresponding series is challenging.

Matrix element and the transition probability for Majorana neutrinos

For two Majorana neutrinos, the matrix element reads

$$\mathcal{M}_{e\to\mu} \propto \kappa_{-}^{\dagger}(p_\mu)\left[\sin 2\theta \frac{1}{2}(\Sigma_{22} - \Sigma_{11}) + \cos 2\theta \Sigma_{12}\right]\kappa_{-}(p_e)$$

One should compute the 3D Fourier transform

$$\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \Sigma_{ab}(E, \mathbf{q}) e^{\mathrm{i}\mathbf{q}\mathbf{L}}$$

The final form of the matrix element

$$\mathcal{M}_{e\to\mu} \propto -\frac{\left(\frac{\Delta m^2}{4E} + \frac{g_2 - g_1}{2}\right)\sin 2\theta + g\cos 2\theta}{\sqrt{\left(\frac{\Delta m^2}{4E} + \frac{g_2 - g_1}{2}\right)^2 + g^2}} \sin\left(\sqrt{\left(\frac{\Delta m^2}{4E} + \frac{g_2 - g_1}{2}\right)^2 + g^2} |L|\right)$$

Effective potentials of the neutrino matter interaction in the mass basis

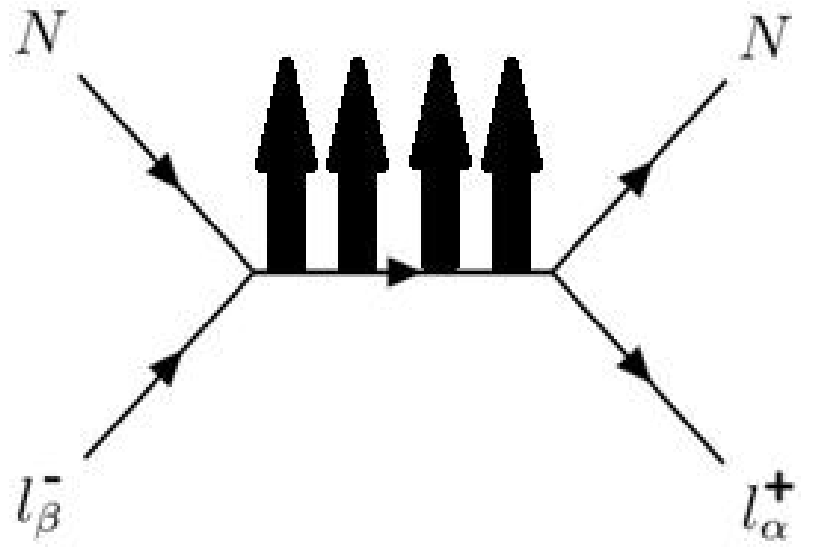
$$g_1 = V_e \cos^2\theta + V_\mu \sin^2\theta, \quad g_2 = V_e \sin^2\theta + V_\mu \cos^2\theta, \quad g = (V_e - V_\mu)\sin\theta\cos\theta$$

One gets the transition probability for the neutrino flavor oscillations in matter which coincides with the prediction of the QM approach

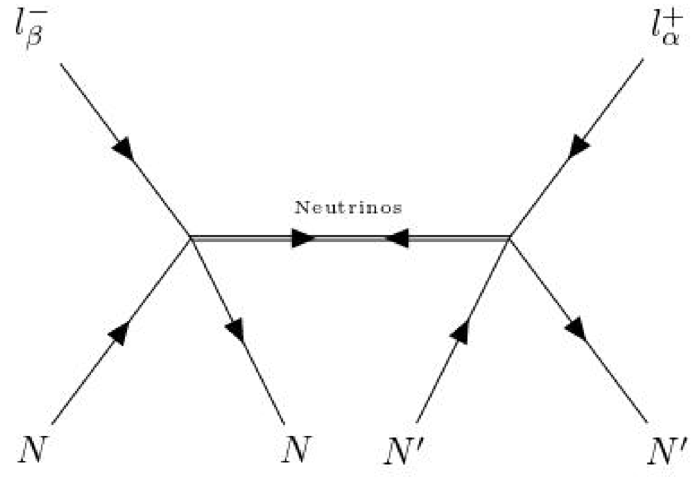
$$P_{e\to\mu} = \frac{\left(\frac{\Delta m^2}{4E}\sin 2\theta\right)^2}{\left(\frac{\Delta m^2}{4E}\sin 2\theta\right)^2 + \left(\frac{\Delta m^2}{4E}\cos 2\theta - \frac{G_\mathrm{F} n_e}{\sqrt{2}}\right)^2} \sin^2\left(\sqrt{\left(\frac{\Delta m^2}{4E}\sin 2\theta\right)^2 + \left(\frac{\Delta m^2}{4E}\cos 2\theta - \frac{G_\mathrm{F} n_e}{\sqrt{2}}\right)^2} |L|\right)$$

Spin-flavor precession in a magnetic field

- We consider two Majorana neutrinos
- Vaccum mixing angle and masses are nonzero
- Neutrinos have a transition magnetic moment in the mass basis
- Magnetic field is transverse with respect to line connecting a source and a detector



QFT description of the spin-flavor precession



Feynman diagram corresponding to the spin-flavor precession involves the a lepton and an anti-lepton

$$S = -\frac{1}{2}\left(\sqrt{2}G_{\text{int}}\right)^2 \int \mathrm{d}^4x\mathrm{d}^4y \left\langle l_\alpha^+, N, N' | T\{j^\mu(x)J_\mu(x)j^\nu(y)J_\nu(y)\} | l_\beta^-, N, N' \right\rangle$$

S-matrix element

The rest of the quantities is the same as in the description of neutrino flavor oscillations in matter

Matrix element for the spin-flavor precession

Matrix element is computed analogously to oscillations in matter. It has the form

$$\mathcal{M}_{\beta\to\bar{\alpha}} = \sum_{ab} U^*_{\alpha a} U^*_{\beta b} \boldsymbol{\kappa}^\dagger_{\alpha +}(p_\alpha) \left(\int \frac{\mathrm{d}^3 q}{(2\pi)^3} \Sigma_{ab}(E, \mathbf{q}) e^{\mathbf{iqL}} \right) \boldsymbol{\kappa}_{\beta -}(p_\beta)$$

$$\Sigma_{ab}(x-y) = \left\langle 0 | T \left\{ \xi_a(x) \eta^\dagger_b(y) \right\} | 0 \right\rangle$$

Propagators of mass states

“Neutrino” wavefunction $\psi_a = \begin{pmatrix} \mathrm{i}\sigma_2 \eta^*_a \\ \eta_a \end{pmatrix}$

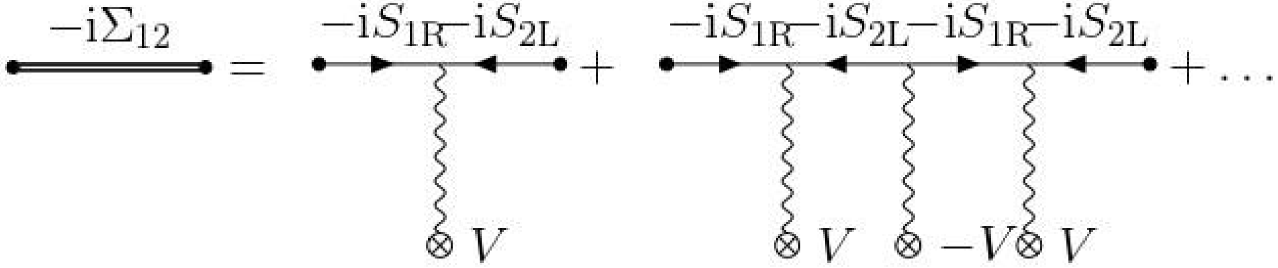
“Antineutrino” wavefunction $\psi_a = \begin{pmatrix} \xi_a \\ -\mathrm{i}\sigma_2 \xi^*_a \end{pmatrix}$

In the absence of a magnetic field, the spin-flavor precession is analogous to neutrino-antineutrino oscillations. In vacuum, the propagators corresponding to such oscillations are diagonal. This is why the propagators here contain both "neutrino" and "antineutrino" contributions.

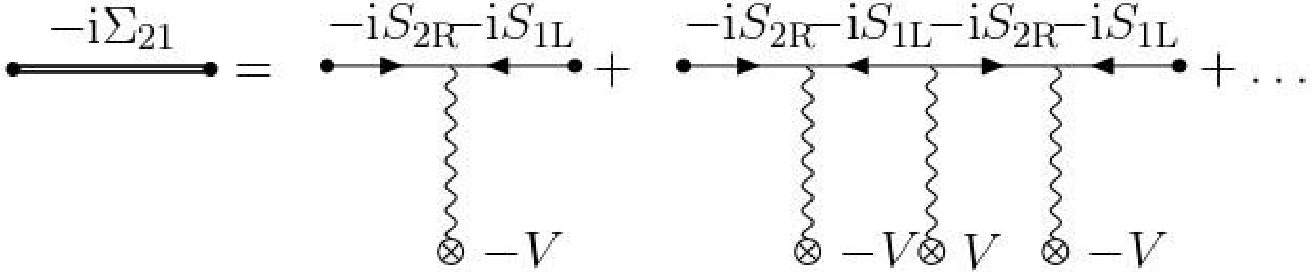
Propagators of mass eigenstates in a magnetic field

Only Σ_{12} и Σ_{21} contribute to the matrix element

Feynman diagrams



$V = \mathrm{i}\mu(\sigma\mathbf{B})$



Perturbative series

$$-\mathrm{i}\Sigma_{12} = (-\mathrm{i}S_{1\mathrm{R}})V(-\mathrm{i}S_{2\mathrm{L}}) + (-\mathrm{i}S_{1\mathrm{R}})V(-\mathrm{i}S_{2\mathrm{L}})(-V)(-\mathrm{i}S_{1\mathrm{R}})V(-\mathrm{i}S_{2\mathrm{L}}) + \cdots$$

$$-\mathrm{i}\Sigma_{21} = (-\mathrm{i}S_{2\mathrm{R}})(-V)(-\mathrm{i}S_{1\mathrm{L}}) + (-\mathrm{i}S_{2\mathrm{R}})(-V)(-\mathrm{i}S_{1\mathrm{L}})V(-\mathrm{i}S_{2\mathrm{R}})(-V)(-\mathrm{i}S_{1\mathrm{L}}) + \cdots$$

Dyson equation $\mathrm{i}\Sigma_{12}^{-1} = -(S_{1\mathrm{R}}VS_{2\mathrm{L}})^{-1} + V, \quad \mathrm{i}\Sigma_{21}^{-1} = (S_{2\mathrm{R}}VS_{1\mathrm{L}})^{-1} - V$

Vacuum propagators (thin lines) $S_{a\mathrm{L,R}}(p) = \dfrac{\mathrm{i}}{2}\left[1 \mp \dfrac{(\sigma\mathbf{p})}{E_a}\right]\dfrac{1}{p_0 - E_a + \mathrm{i}0}$

Solution of the Dyson equation for ultrarelativistic neutrinos

While calculating the propagators, we take that $p/E_{1,2} < 1$ to avoid the divergencies in finding the reciprocal vaccum propagators. We put $p/E_{1,2} \to 1$ in the final expression

$$\Sigma_{12} = -\Sigma_{21} = -\frac{\mu(\sigma \cdot [\mathbf{B} + \mathrm{i}(\hat{q} \times \mathbf{B}) - (\hat{q}\mathbf{B})\hat{q}])}{2[(E - E_1 + \mathrm{i}0)(E - E_2 + \mathrm{i}0) - \mu^2 B^2 + \mu^2 (\hat{q}\mathbf{B})^2]}$$

Since neutrinos are virtual particles, ***q*** can be arbitrary. In fact, we calculate the 3D integral d^3q. Thus, one cannot drop the term $\propto (\hat{q}B)$ despite the magnetic field is transverse ($\boldsymbol{B} \perp \boldsymbol{L}$). Moreover, these terms are comparable with other terms in the propagator.

Matrix element and the transition probability for two neutrinos

In computation of matrix element, we the QFT terms in the propagator

$$\mathcal{M}_{e\to\bar{\mu}} = -\frac{\mathrm{i}Ee^{\mathrm{i}EL}}{2\pi L}\frac{\mu B}{\mathfrak{E}}\left(1-\frac{(\mu B)^2}{2\mathfrak{E}E}\right)\left(\sin\mathfrak{E}L - \mathrm{i}\frac{\mathfrak{E}}{E}\cos\mathfrak{E}L\right) \quad \mathfrak{E} = \sqrt{(\mu B)^2 + \left(\frac{\Delta m^2}{4E}\right)^2}$$

Transition probability

$$P_{\nu_e\to\bar{\nu}_\mu}(L) = P_{\max}\sin^2\left(\sqrt{(\mu B)^2 + \left(\frac{\Delta m^2}{4E}\right)^2} L\right)$$

$$P_{\max}^{(\mathrm{qm})} = \frac{(\mu B)^2}{(\mu B)^2 + \left(\frac{\Delta m^2}{4E}\right)^2}, \quad P_{\max}^{(\mathrm{qft})} = -\frac{(\mu B)^4}{E\left[(\mu B)^2 + \left(\frac{\Delta m^2}{4E}\right)^2\right]^{3/2}}$$

Small correction to the amplitude of the transition $|P_{\max}^{(\mathrm{qft})}|/P_{\max}^{(\mathrm{qm})} \sim \mu B/E$ probability arises from the QFT terms in the dressed propagators. One can neglect it for ultrarelativistic neutrinos

Charged leptons with arbitrary energies

We assume that an left-polarized electron moves along ***L***, whereas right-polarized anti-muon can move in arbitrary direction. The chiral projections of their wavefunctions are

$$\kappa_{e\mathrm{L}} = \sqrt{\frac{1+v_e}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \kappa_{\bar{\mu}\mathrm{R}} = -\sqrt{\frac{1+v_{\bar{\mu}}}{2}} \begin{pmatrix} e^{-\mathrm{i}\phi_{\bar{\mu}}} \cos\frac{\theta_{\bar{\mu}}}{2} \\ e^{\mathrm{i}\phi_{\bar{\mu}}} \sin\frac{\theta_{\bar{\mu}}}{2} \end{pmatrix}$$

The measurable quantity is the total cross-section

$$\sigma_{e\to\bar{\mu}} = \frac{E^4 G_{\mathrm{int}}^4}{2\pi L^2} P_{\nu_e\to\bar{\nu}_\mu}^{(\mathrm{eff})}, \quad P_{\nu_e\to\bar{\nu}_\mu}^{(\mathrm{eff})} = f(v_e, v_{\bar{\mu}}) P_{\nu_e\to\bar{\nu}_\mu}^{(\mathrm{qm})}, \quad f(v_e, v_{\bar{\mu}}) = (1+v_e)(1+v_{\bar{\mu}})\frac{v_{\bar{\mu}}}{4v_e}$$

The effective transition probability oscillates is space. It is less than the QM transition probability since $f<1$.

Conclusion

- Major assumptions are
  - ✓Ultrarelativity of neutrinos simplifies the derivation of dressed propagators and the computation of 3D Fourier transform
  - ✓Spatial homogeneity of external fields
  - ✓Consideration of two mass eigenstates. Perturbative series become branching for greater number of neutrinos
- QFT terms in the dressed propagators result in the small correction to the transition probability of the spin-flavor precession, which can be neglected for ultrarelativistic neutrinos
- Accounting for external fields is an essentially nonperturbative phenomenon if neutrino oscillations are described in frames of QFT. One has to sum all the terms in the perturbation series to get the correct result for the transition probability