Entanglement entropy in QCD under extreme condition

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Outlook

- Entanglement Entropy (EE)
  - Quantum information
  - QFT
- Measurement of EE
  - In solid state physics
  - In high energy — Challenges in Direct Measurement
    - Heavy Ion Collisions (HIC). HICs produce thousands of particles, making quantum state trucking impossible
    - Other proposal to measure entanglement in HE: deep inelastic scattering; jet production
- Calculation of EE in holography

Entanglement Entropy

- Quantum system is described by an algebra $\mathfrak{U}$ of observables and the quantum state ρ – positive linear functional on it.
- According GNS construction the algebra $\mathfrak{U}$ has a representation a Hilbert space $\mathcal{H}$.
- The state ρ on algebra of operators in the Hilbert space often can be given as a trace
$$\rho(O) = \mathrm{Tr}[\rho \cdot O]$$
- von-Neumann entropy of ρ: $S(\rho) = -\mathrm{Tr}[\rho \ln \rho]$
- For the composed system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ a separable state is such that:
$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$$
Entangled state $|\Psi\rangle \in \mathcal{H}$ if it is not separable
- Reduced density matrix ρ_A for the subsystem A is obtained by tracing out with respect to $\mathcal{H}_B$ by
$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}[\rho]$$
- The entanglement entropy is defined as the von-Neumann entropy for ρ_A
$$S_A = -\mathrm{Tr}[\rho_A \ln \rho_A]$$

EE in Lattice Gauge Theories

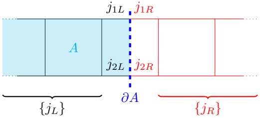
P.Buividovich, M.Polikarpov, "Numerical study of entanglement entropy in SU(2) lattice gauge theory," Nucl. Phys. B 802 (2008) 458

L. Ebner *et al.* "Entanglement Properties of SU(2) Gauge Theory," arXiv:2411.04550

- The Kogut-Susskind (1974) Hamiltonian

$$H = \frac{g^2}{2}\sum_{\mathrm{L}}(E_i^a)^2 - \frac{2}{a^2 g^2}\sum_{\mathrm{P}}\mathrm{Tr}\left[\prod_{(\boldsymbol{n},\hat{i})\in \mathrm{P}} U(\boldsymbol{n},\hat{i})\right]$$

- $\rho = \dfrac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}$



Plot from 2411.04550

Prediction of quantum entanglement in particle jets

A. Florio *et al.* "Real-Time Nonperturbative Dynamics of Jet Production in Schwinger Model: Quantum Entanglement and Vacuum Modification," PRL 131(2023) 021902

- **This prediction lays groundwork for experimental tests of entanglement at particle colliders**

Y. Afik,*et al.* "Quantum Information meets High-Energy Physics: Input to the update of the European Strategy for Particle Physics," 2504.00086

- "Some of the most astonishing and prominent properties of Quantum Mechanics, such as entanglement and Bell nonlocality, have only been studied extensively in dedicated low-energy laboratory setups".

- "The feasibility of these studies in the high-energy regime explored by particle colliders was only recently shown, and has gathered the attention of the scientific community".

The goal of this talk study behavior of

entanglement entropy in QCD using holographic methods.

Holographic QCD - phenomenological approach

- Perturbation methods are not applicable to describe QCD phase diagram
- Lattice methods do not work, because of problems with the chemical potential.
- AdS/CFT [What is wrong with exact AdS/CFT applications to QCD]
- Holographic QCD - phenomenological model(s)

- One of goals of Holographic QCD – describe QCD phase diagram

- Requirements:
  - reproduce the QCD results from perturbation theory at short distances
  - reproduce Lattice QCD results at large distances (~ 1 fm) and small μ_B

Holographic QCD vs exact AdS/CFT

Maldacena,1998

What is wrong with exact AdS/CFT applications to QCD:

- **QCD is not conformal, conformal invariance is restored only in high energy**

Holographic QCD vs exact AdS/CFT

Maldacena,1998

What is wrong with exact AdS/CFT applications to QCD:

- **QCD is not conformal, conformal invariance is restored only in high energy**
- **No confinement in $BHAdS_5$**

Holographic model of an anisotropic plasma in a magnetic field at a nonzero chemical potential

I.A, K. Rannu, P.Slepov, JHEP, 2021

$$S=\int d^5x\,\sqrt{-g}\left[R-\frac{f_1(\phi)}{4}F_{(1)}^2-\frac{f_B(\phi)}{4}F_{(B)}^2-\frac{1}{2}\partial_M\phi\partial^M\phi-V(\phi)\right]$$

$$ds^2=\frac{L^2}{z^2}\,\mathfrak{b}(z)\left[-g(z)\,dt^2+dx^2+dy_1^2+e^{c_Bz^2}dy_2^2+\frac{dz^2}{g(z)}\right]$$

$A_{(1),m}=A_t(z)\delta_m^0,\ A_t(0)=\mu, \qquad F_{(B)}=dx\wedge dy^1$

Giataganas'13; IA, Golubtsova'14; Gürsoy, Järvinen '19; Dudal et al.'19

$$\mathfrak{b}(\mathbf{z})=\mathbf{e}^{\mathbf{2}\mathcal{A}(\mathbf{z})}\ \Leftrightarrow\ \text{quarks mass}$$

"Bottom-up approach"

Heavy quarks (b, t):

$\mathcal{A}(z)=-cz^2/4$ *Andreev, Zakharov'06*

$\mathcal{A}(z)=-cz^2/4+pz^4$ *IA, Hajilou, Rannu, Slepov, EPJ C (2023)83*

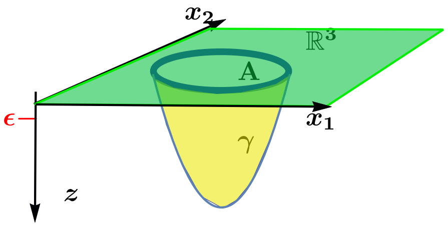
Light quarks (d, u)

$\mathcal{A}(z)=-a\ln(bz^2+1)$ *Li, Yang, Yuan'17*

φ - dilaton, $\alpha(z)=e^{\varphi(z)}$ - **running** coupling in HQCD

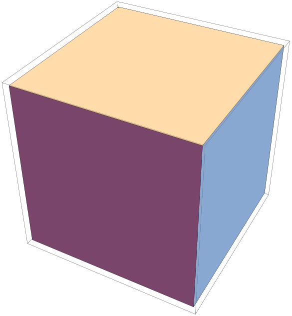
The Ryu Takayanagi prescription :
$S_{HEE}(A)$ for spatial 3-dim domain A with boundary ∂A, is obtained by extremizing the volume of the static 3-dim domain γ, which is located in the 5-dim space $\mathcal{M}$ (AdS_5 or its deformations) and on the boundary of the 5-dim space $\partial\mathcal{M}$ coincides with ∂A

$$S_{HEE}(A) = \frac{1}{4G_{5N}} \min_{\gamma} \int_{\gamma} d^3\xi \sqrt{|\det \mathcal{G}_{s,MN}\partial_\alpha x^M \partial_\beta x^N|}$$

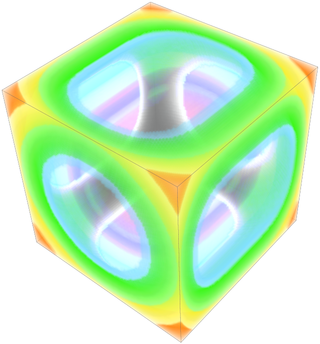


$$\partial\gamma\Big|_{\partial\mathcal{M}} = \partial A$$

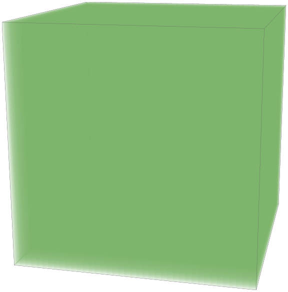
Visualizing the Entanglement Volume



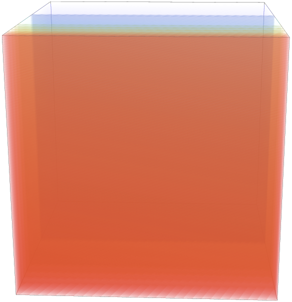
$$f(x_1, x_2, x_3) = x_1^2 + x_2^4 + x_3^2$$



HEE for a slab-shaped region.



$$f(x_1, x_2, x_3) = x_3^2$$



Orientation of the entanglement region vs HIC geometry

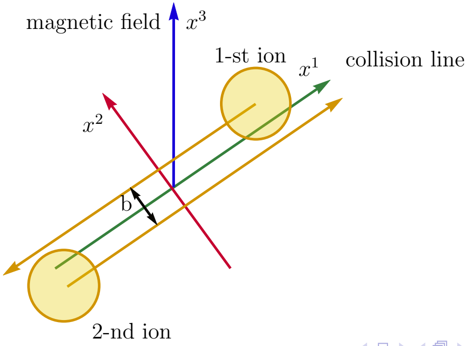
Schematic picture of two ions collisions

Natural coordinate system:

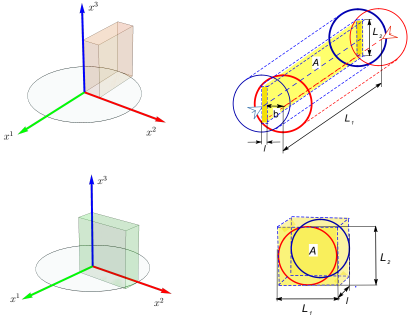
x^1 (longitudinal axis x) along the line of collision

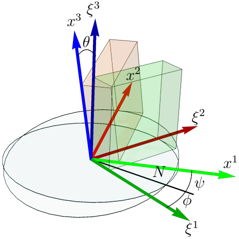
x^2 (1-st transversal axis y_1) along of the impact parameter.

x^3 (2-nd transversal axis y_2) along of magnetic field



Orientation of the entanglement region (slab) vs HIC geometry

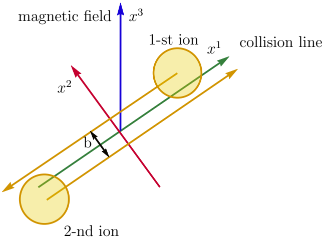




Orientation of the entangled region is set by Euler angles

General anisotropic holographic model associated with HIC geometry

Natural coordinate system:
x^1 (longitudinal axis x) along the line of collision
x^2 (1-st transversal axis y_1) along of the impact parameter.
x^3 (2-nd transversal axis y_2) along of magnetic field



General anisotropic holographic model

$$ds^2 = \frac{L^2 b_s(z)}{z^2} \sum_{M=0}^{4} G_M(z)(dX^M)^2$$

$$X^0 = t,\, X^1 = x,\, X^2 = y_1,\, X^3 = y_2,\, X^4 = z$$

$$G_0 = -g(z), \quad G_4 = \frac{1}{g(z)}$$

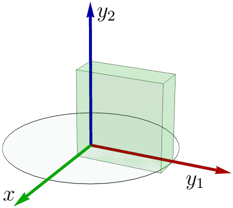
$$G_i = \mathfrak{g}_i(z),\, i = 1, 2, 3,$$

$b_s(z) = b(z) e^{\sqrt{\frac{2}{3}}\phi(z)}$ AdS deformation factor
$g(z)$ blackening function
$\mathfrak{g}_i(z)$ anisotropy factors.

EE for subsystem A_{xYY} allocated along x-direction

I.A., Phys. Part. Nuclei Lett. 16, 486 (2019)
I.A., A.Patrushev, P.Slepov, JHEP (2020)07, 043

$$x \in [0, |l_x| << L_x], \quad y_1 \in [0, L_{y_1}], \quad y_2 \in [0, L_{y_2}]$$

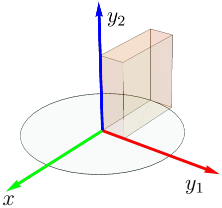


$$\mathcal{S}_{xYY} = \frac{\mathcal{A}_{xYY}}{L_{y_1} L_{y_2}} = \int_0^{z_*} \frac{b_s^{3/2}(z)}{z^{1+2/\nu}} \sqrt{1 + \frac{z'^2}{g(z)}} dx$$

$$b_s(z, \nu) \equiv e^{cz^2/2 + \sqrt{\frac{2}{3}}\phi(z, z_h, \nu)}$$

EE for subsystem A_{yXY} allocated along y_1-direction

$$x \in [0, L_x], \quad y_1 \in [0, |l_{y_1}| << L_{y_1}], \quad y_2 \in [0, L_{y_2}]$$



$$\mathcal{S}_{yXY} = \int_0^{z*} \frac{b_s^{3/2}(z)}{z^{1+2/\nu}} \sqrt{1 + \frac{z'^2}{g(z)\, z^{2-2/\nu}}} dx$$

BI-action:

$$\mathcal{S} = \frac{T}{2\pi\alpha} \int_{-\ell}^{\ell} M(z) \sqrt{\mathcal{F}(z) + z'^2} dx, \qquad \mathcal{V}(z) = M(z)\sqrt{\mathcal{F}(z)}$$

y_1-direction is equivalent to y_2 without magnetic field

Renormalization (*)

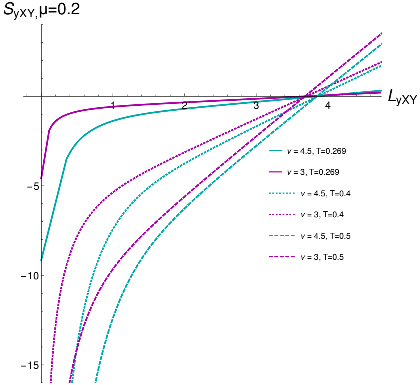
For xYY case and $1 \le \nu \le 1.67$ we have to perform renormalization (just one substraction):

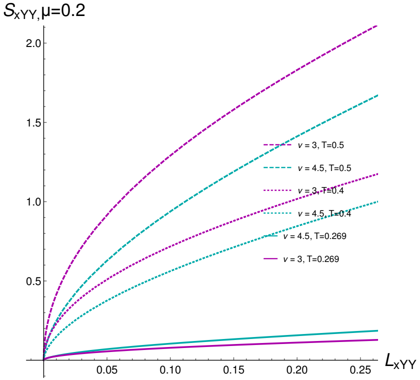
$$\frac{1}{2}\mathcal{S}_{xYY,ren} = \int_{\epsilon}^{z_*} dz \left[\frac{b_s^{3/2}(z)}{z^{1+2/\nu}} \frac{1}{\sqrt{g(z)(1-\frac{\mathcal{V}^2_{xYY}(z_*)}{\mathcal{V}^2_{xYY}(z)})}} - \frac{b_{s,as}^{3/2}(z)}{z^{1+2/\nu}} \right] + \int^{z_*} \frac{b_{s,as}^{3/2}(z)}{z^{1+2/\nu}} dz$$

For xYY case and $\nu > 1.67$ we have an integrable singularity
For yXY case and $\nu \ge 1$ we have nonintegrable singularity and have to perform renormalization (just one substraction):

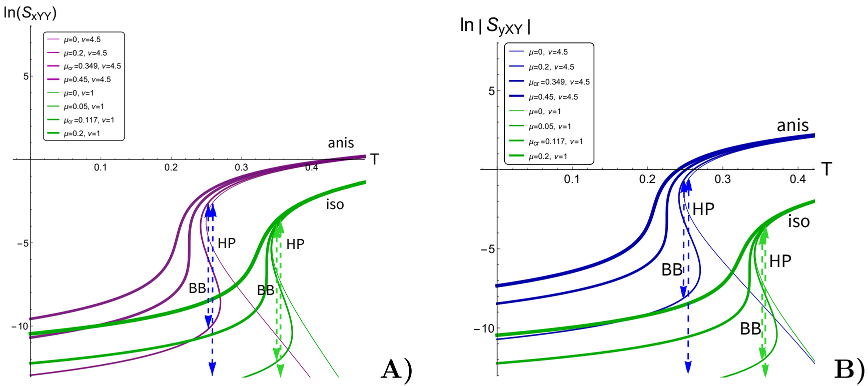
$$\frac{1}{2}\mathcal{S}_{yXY,ren} = \int_{\epsilon}^{z_*} dz \left[\frac{b_s^{3/2}(z)}{z^{2+1/\nu}} \frac{1}{\sqrt{g(z)(1-\frac{\mathcal{V}^2_{yXY}(z_*)}{\mathcal{V}^2_{yXY}(z)})}} - \frac{b_{s,as}^{3/2}(z)}{z^{2+1/\nu}} \right] + \int^{z_*} \frac{b_{s,as}^{3/2}(z)}{z^{2+1/\nu}} dz$$

Numerical Results: HEE dependence on length (*)





Numerical Results: HEE dependence on T



HEE of the slab with $\ell = 1$: A) longitudinal B) transversal

Isotropic: $\nu = 1,\ \mu_{cr} = 0.117,\ T_{cr} = 0.33$

Anisotropic: $\nu = 4.5,\ \mu_{cr} = 0.349,\ T_{cr} = 0.33$

Entanglement Entropy Density η

$$\eta = \frac{dS(\ell)}{d\ell}$$

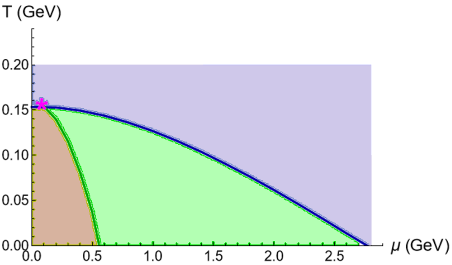
The advantage of dealing with the HEE density is that it has no divergences.

$$\eta(z_*) = \frac{dS(z_*)}{d\ell(z_*)} = \frac{\mathcal{V}(z_*)}{4}$$

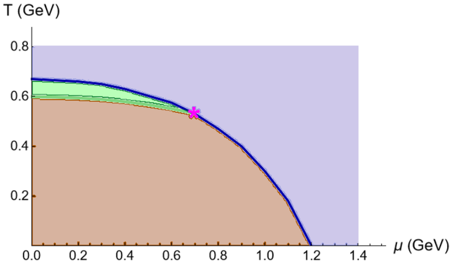
General expression for full anisotropic case

$$\eta(z_*) = \frac{1}{4}\frac{L^3}{z_*^3} b_s^{3/2}(z_*)(\mathfrak{g}_1(z_*)\mathfrak{g}_2(z_*)\mathfrak{g}_3(z_*))^{1/2}$$

1-st order phase transition in HQCD.



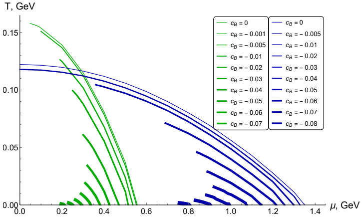
LQ



HQ

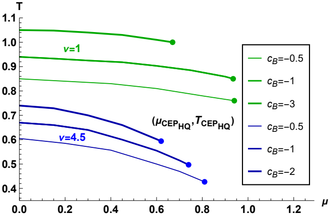
1-st order phase transition in HQCD, $B \neq 0$

Light quarks



I.A, Ermakov, Rannu, Slepov,EPJC'23

Heavy quarks



I.A, A. Hajilou, K.R., P.S.EPJC'23

Conclusion

- **Entanglement entropy serves as a powerful diagnostic tool in holographic QCD. We have shown that it exhibits a clear jump at first-order phase transitions (FOPT).**

- **This sharp signature provides a robust method for locating the position of FOTR in the (T, μ)-plane**

- **Looking forward, we propose a compelling connection to experiment: the entanglement entropy of the collision region in heavy-ion collisions may be directly identified with the final-state particle multiplicity. This provides a potential bridge between our theoretical framework and experimental observables**

Other theoretical methods for locating of FOTR in talks: P. Slepov, M.Usova

Thank you!