Synchrotron radiation in odd--dimensional electrodynamics

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Green's functions

Momentum representation:

$$G_{\mathrm{ret/adv}}(k) = -\frac{1}{k^2 \pm i\epsilon\omega}\,,$$

Symmetric and Radiative GF:

$$G_{\mathrm{sym/rad}}(k) = \frac{1}{2}\Big[G_{\mathrm{ret}}(k) \pm G_{\mathrm{adv}}(k)\Big]\,.$$

Coordinate representation:

$$G_{\mathrm{ret/adv}}(X) = 2\,\theta(\pm T)\,G_{\mathrm{sym}}(X^2)\,, \qquad G_{\mathrm{rad}}(X) = \mathrm{sgn}(T)\,G_{\mathrm{sym}}(X^2)$$

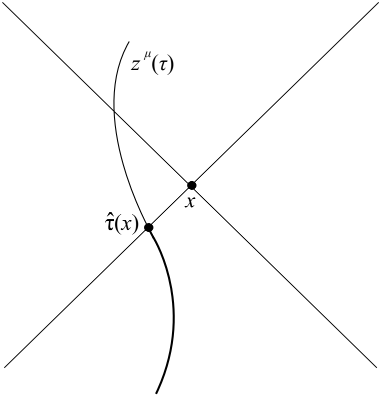
Even dimensions:

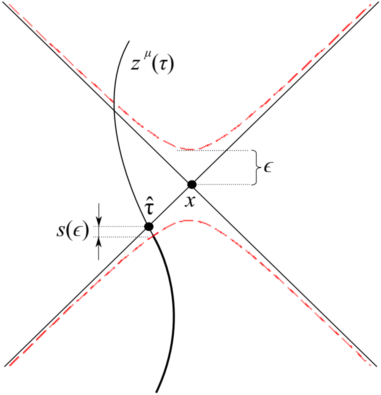
$${}^{D}G_{\mathrm{ret}}(X) = \frac{\theta(T)}{2\pi^{D/2-1}}\,\delta^{D/2-2}(y) \qquad y = (x-x')^2$$

Odd dimensions:

$${}^{D}G_{\mathrm{ret}}(X) = \frac{\theta(t-t')}{2\pi}\left(\frac{1}{\pi}\frac{\partial}{\partial y}\right)^{\frac{D-3}{2}}\frac{\theta(y)}{\sqrt{y}} \qquad y = (x-x')^2$$

Retarded potentials





Regularisation:

$$X^2 \longrightarrow X^2 - \epsilon^2 \qquad\qquad \epsilon \to 0^+$$

Lienard–Wiechert potentials:

$$^D\varphi_{\text{ret}} = {}^D G_{\text{ret}} * j \,.$$

Even space-time dimensions

Lorentz–Dirac equations:

$$f_{\mathrm{LD}} \equiv \mathrm{f.p.}\, f_{\mathrm{ret}} = \frac{f_{\mathrm{ret}} - f_{\mathrm{adv}}}{2} \equiv f_{\mathrm{rad}}\,,$$

$$f_{\mathrm{sym}} = \mathrm{i.p.}\, f_{\mathrm{ret}} = \frac{f_{\mathrm{ret}} + f_{\mathrm{adv}}}{2}$$

Energy-momentum splitting:

$$T^{\mu\nu} = \underbrace{A\frac{c^\mu c^\nu}{\varrho^{D/2-1}}}_{\mathrm{emit}} + \underbrace{\frac{a^{\mu\nu}}{\varrho^{D/2}} + \ldots + \frac{s^{\mu\nu}}{\varrho^{D}}}_{\mathrm{bound}}$$

LD equation in (3+1)D for EM field

$$f^\mu_{\mathrm{ret}} = \frac{e_4^2}{6\pi}\Bigg[\underbrace{\overbrace{-\frac{3}{4\epsilon}}^{\mathrm{sym}} + \overbrace{\dddot{z}^\mu}^{\mathrm{rad}}\!\!\!\!}_{\mathrm{bound}}\overbrace{+ \underbrace{\ddot{z}^2\dot{z}^\mu}_{\mathrm{emit}}}\Bigg] + f^\mu_{\mathrm{ext}}\,,$$

$$f^\mu_{\mathrm{ret}} = \underbrace{\overbrace{f^\mu_{\mathrm{sym}}}^{\mathrm{singular}} + f^\mu_{\mathrm{rr}}}_{\mathrm{bound}} \overbrace{+ \underbrace{f^\mu_{\mathrm{emit}}}_{\mathrm{emit}}}^{\mathrm{rad\,(finite)}} + f^\mu_{\mathrm{ext}}$$

Singular part ϵ^{-1} $(\epsilon \to 0^+)$ $\longrightarrow$ mass renormalization

Synchrotron motion

LD equation in (5+1)D: ($v = \dot{z}, a = \ddot{z}, \ddot{a} = \dddot{z}$,etc.)

$$f^{\mu}_{\text{emit}} = -\frac{e_6^2}{4\pi^2}\left(\frac{1}{3}\left(a^2\right)^2 v^{\mu} + \frac{1}{35}\, a^2 \dot{a}^{\mu} - \frac{1}{7}\left(a\dot{a}\right) a^{\mu} - \frac{2}{15}\,\dot{a}^2 v^{\mu}\right),$$

$$f^{\mu}_{\text{rr}} = -\frac{e_6^2}{30\pi^2}\left(\dddot{a}^{\mu} + \frac{16}{7}\, a^2 \dot{a}^{\mu} + \frac{60}{7}\left(a\dot{a}\right) a^{\mu} + 4\,\dot{a}^2 v^{\mu} + 4\left(a\ddot{a}\right) v^{\mu}\right)$$

Emission rate I:

$$(3+1): \qquad I_4 = \frac{e_4^2 w_0^2}{6\pi}\,\gamma^2 v^2$$

$$(5+1): \qquad I_6 = \frac{e_6^2 w_0^4}{12\pi^2}\left(\gamma^2 v^2 + \frac{2}{5}\right)\gamma^2 v^2$$

$$(7+1): \qquad I_8 = \frac{e_8^2 w_0^6}{72\pi^3}\left(5\gamma^4 v^4 + \frac{346}{105}\,\gamma^2 v^2 + \frac{9}{35}\right)\gamma^2 v^2$$

where cyclotron frequencies

$$H := F^{xy}, \qquad \gamma v \equiv \gamma_* , \qquad w_0 \equiv \frac{e_D H}{m} \qquad w \equiv \frac{e_D H}{m\gamma}$$

Even dimensions

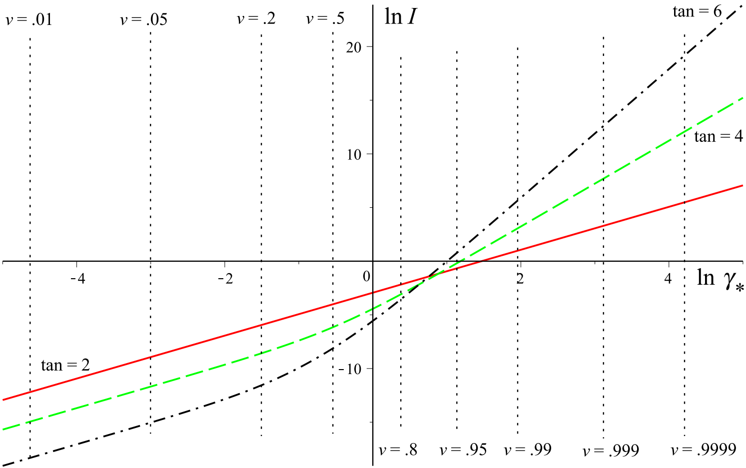
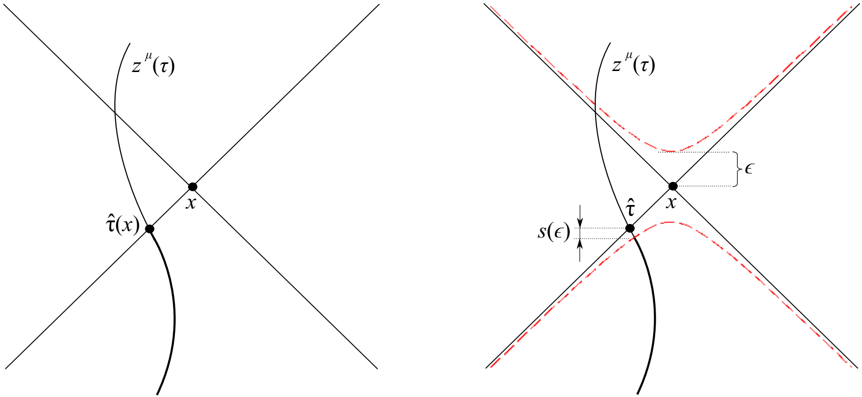


Figure: Plots of electromagnetic synchrotron radiation rate versus γ_* in doubly logarithmic mode for $D = 4$ (red, solid), $D = 6$ (green, dashed) and $D = 8$ (black, dashdotted), in units $e = w_0 = 1$.

Odd dimensions



No known LD equation

- Kazinski P. O., Lyakhovich S. L., Sharapov A. A., Phys.Rev. D. **66** (2002) 025017
- Galtsov D. V., Phys.Rev. D. **66** (2002) 025016
- Y.Yaremko (2000 – 2010's), with extra lengthy parameters

Logarithmic IR divergence when retarded GF used (for arbitrary motion)!

Harmonic analysis

Synchrotron radiation:

- 3D: E.Shuryak, H-U. Yee, I.Zahed, Phys.Rev. D **85** (2012) 104007. **66** (2002) 025017
- 5D: Galtsov D. V., M. Khlopunov, ArXiV: 2003.00261[hep-th]

Fourier-transform:

$$\omega_m = mw\,, \qquad k_m^\mu = wm(1, \boldsymbol{n})\,.$$

Generic formula for emission rate:

$$\frac{d}{d\Omega}\begin{Bmatrix} I^{\rm sc} \\ I^{\rm em} \end{Bmatrix} = \frac{w^{D-2}}{2(2\pi)^{D-2}} \sum_{m=1}^{\infty} m^{D-2} \begin{Bmatrix} |j_m(\boldsymbol{k})|^2 \\ -j_m^*(\boldsymbol{k}) \cdot j_m(\boldsymbol{k}) \end{Bmatrix},$$

Angular distributions

$$\frac{d}{d\theta}\begin{Bmatrix} I^{\rm sc} \\ I^{\rm em} \end{Bmatrix} = \frac{w^{D-2}\cos\theta\,\sin^{D-4}\theta}{(2\sqrt{\pi})^{D-3}\Gamma\left(\frac{D-3}{2}\right)} \sum_{m=1}^{\infty} m^{D-2} \begin{Bmatrix} q^2\gamma^{-2} J_m^2(mz) \\ e^2\left[v^2 {J'_m}^2(mz) + \tan^2\theta\, J_m^2(mz)\right] \end{Bmatrix}$$

where $z = v\cos\theta$

Kapteyn series of 2nd kind:

$$C_\lambda(z) := \sum_{m=1}^{\infty} m^\lambda J_m^2(mz), \qquad \tilde{C}_\lambda(z) := \sum_{m=1}^{\infty} m^\lambda \left(J'_m(mz)\right)^2,$$

For even λ (even D) these series are known in closed form!

Effective number of discrete modes

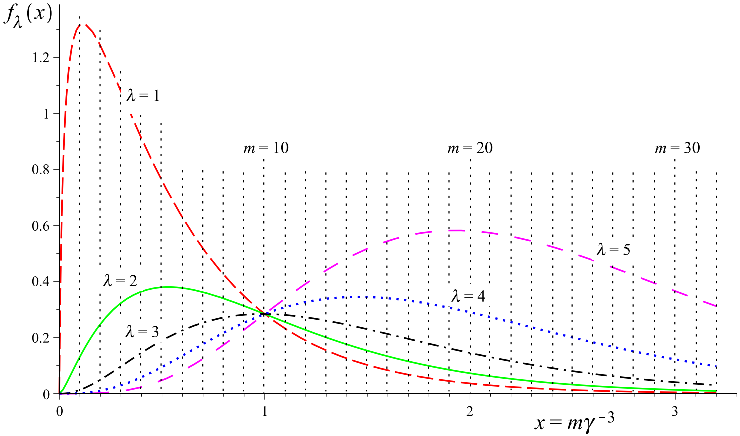


Figure: Plots of continuous $f_\lambda(x) = x^\lambda J^2_{x\gamma^3}(x\gamma^3 v)$ versus $x = m\gamma^{-3}$ (rescaled by γ^6) for $v = 0.886$ ($\gamma = 2.154$) for $\lambda = 1$ (red, dashed), $\lambda = 2$ ((green, solid), $\lambda = 3$ (black, dashdotted), $\lambda = 4$ (blue, dotted) and $\lambda = 5$ (magenta, spacedashed). True discrete points of Kapteyn series, corresponding to natural m, lie on dashed vertical lines.

Odd dimensions

Even $\lambda = 2s$:

$$C_{2s}(z) = \frac{z^2 P_{2s-1}(z^2)}{(1-z^2)^{3s+1/2}}\,, \qquad \tilde{C}_{2s}(z) = \frac{\tilde{P}_{2s-1}(z^2)}{(1-z^2)^{3s-1/2}}\,,$$

$P_\nu(x)$ and $\tilde{P}_\nu(x)$ — polynomials of degree ν.
Average number of overtones, frequency and angle:

$$\langle m\rangle \sim \langle \delta m\rangle \sim \gamma^3 \qquad \langle\omega\rangle \sim \gamma^3 w \qquad \langle\theta\rangle \sim \gamma^{-1}.$$

Odd dimensions:

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$$C_1(z) = \frac{1}{4\pi}\frac{z^2}{1-z^2}\int\limits_0^\infty \frac{2u - \sin 2u}{(u^2 - z^2\sin^2 u)^{3/2}}\,du$$

$$\tilde{C}_1(z) = \frac{1}{4\pi}\int\limits_0^\infty \frac{\sin 2u - 2u\cos 2u}{(u^2 - z^2\sin^2 u)^{3/2}}\,du$$

plus recurrence for odd $\lambda = \lambda + 2$.

$$u^2 - z^2\sin^2 u \sim (x - x')^2\,, \qquad x = z(\tau)\,, x' = z(\tau - u)$$

Ultra-relativistic regime

Bessel-function approximation when index is close to argument ($z \lesssim 1$)

$$\begin{Bmatrix} J_m(mz) \\ J'_m(mz) \end{Bmatrix} \simeq \frac{\eta}{\pi\sqrt{3}} \begin{Bmatrix} 1 \\ \eta \end{Bmatrix} K_{1/3}\left(\frac{m}{3}\,\eta^3\right) \qquad \eta := \sqrt{1-z^2}$$

Total emitted rate:

$$\begin{Bmatrix} I_D^{\mathrm{sc}} \\ I_D^{\mathrm{em}} \end{Bmatrix} = \frac{\Gamma\left(\frac{D-1}{2}\right)(w\gamma^2)^{D-2}}{2\cdot 3^{\frac{D-2}{2}}\pi^{\frac{D-1}{2}}} \times \begin{Bmatrix} q^2 \\ 2e^2 \end{Bmatrix}, \qquad \left(\frac{I_D^{\mathrm{em}}}{I_D^{\mathrm{sc}}}\right)_{\mathrm{UR}} = 2\,,$$

Frequency distributions: odd D

$$\frac{d}{d\omega}\begin{Bmatrix} I^{\mathrm{sc}} \\ I^{\mathrm{em}} \end{Bmatrix} = \frac{8\,\omega^{D-2} z_{\mathrm{m}}^{-\frac{D+1}{3}}}{3(2\sqrt{\pi}\gamma)^{D+1}(D-4)!!\,w} \begin{Bmatrix} q^2 z_{\mathrm{m}}^{2/3} \\ e^2 \tilde{D}_1 \end{Bmatrix} \tilde{D}^d\left(z_{\mathrm{m}}^{4/3}\left[K_{2/3}^2(z_{\mathrm{m}}) - K_{1/3}^2(z_{\mathrm{m}})\right]\right)$$

where

$$z_{\mathrm{m}} \equiv z\big|_{\theta=0} = \frac{\omega}{3w\gamma^3} \qquad \tilde{D} \equiv \frac{z_{\mathrm{m}}^{1/3}}{2}\frac{\partial}{\partial z_{\mathrm{m}}} z_{\mathrm{m}}^{1/3}\frac{\partial}{\partial z_{\mathrm{m}}} - 2z_{\mathrm{m}}^{2/3}$$

Even less transcendent than for even dimensions!

Conclusions

- Two Kapteyn series' provide unambiguous computation of synchrotron radiation;
- UR and non-relativistic regimes are computable exactly (to the leading order);
- Integral representation of Kapteyn series may shed light to the derivation of odd-dimensional analogues of the LD equation for arbitrary motion;
- Odd-dimensional synchrotron radiation has some common (with even dimensions) and some unique features.

Thank you!