Generalization of equations of relativistic quantum mechanics and the problem of hierarchy of fermionic masses based on the analysis of chains of type mass(-in-mass)^N

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“It seemed to me that the foundation of the work of the mathematical physicist is to get the correct equations, that the interpretation of those equations was only of secondary importance”.

Paul Dirac, Solvay Conference, 1927

“While the accuracy and derivation of equations are foundational for mathematical physicists, interpretation is not necessarily secondary but rather a crucial and often challenging aspect, particularly in quantum mechanics, where the meaning of equations can be debated and different "interpretations" (like the Copenhagen Interpretation) are proposed to understand the reality described by the math. The difficulty in obtaining correct equations is often intertwined with the difficulty of their interpretation, with many physicists arguing that understanding the underlying reality suggested by the equations is just as important as the equations themselves”.

AI Overview, 2025

MECHANICAL EQUIVALENT OF THE (REAL) KLEIN-GORDON EQUATION

In the case of the long-wave approximation (transition to a continuous string)

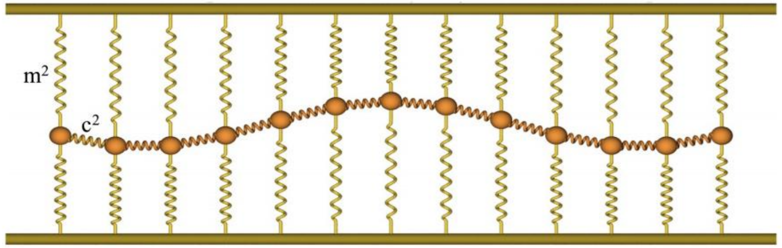


Figure 9.1: Mechanical equivalent of the (real) klein-Gordon equation

Hans de Vries. Understanding Relativistic Quantum Field Theory.
http://physics-quest.org/

← Coupled spring oscillators. The transverse wave

Sergej Faletič

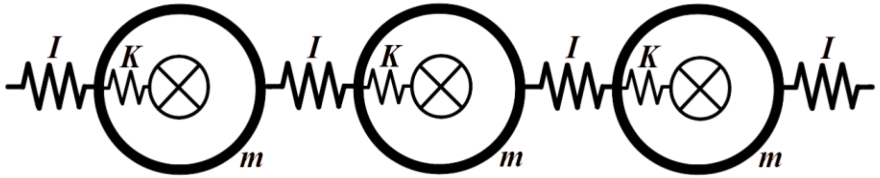
Figure 5. Coupled pendula can be used as a type of braced medium. The medium is made of pendula on strings coupled by a chain hanging from one pendulum to the next (A). On the bottom right side is the driving pendulum (B). A string (C) connects the string of the driving pendulum and one of the strings of the medium, and serves as coupling.

Sergej Faletič. How close can we get waves to wavefunctions, including potential?

Sergej Faletič. The Klein-Gordon string: A tool I've never heard of before

Faculty of mathematics and physics, University of Ljubljana, Slovenia
and Poljane High School, Ljubljana, Slovenia

← Coupled mathematical pendulums. The longitudinal wave



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↑ Coupled spring oscillators. The longitudinal wave

Crawford F.S. Waves. Berkeley Physics Course. McGraw-Hill, New York; 1968. Vol. 3.

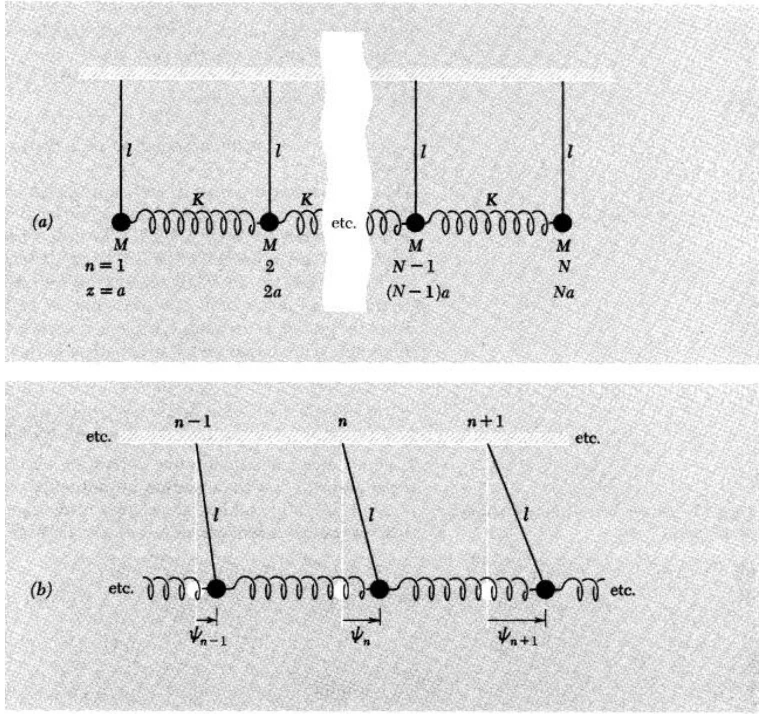


Fig. 2.16 Coupled pendulums. (a) Equilibrium. (b) General configuration.

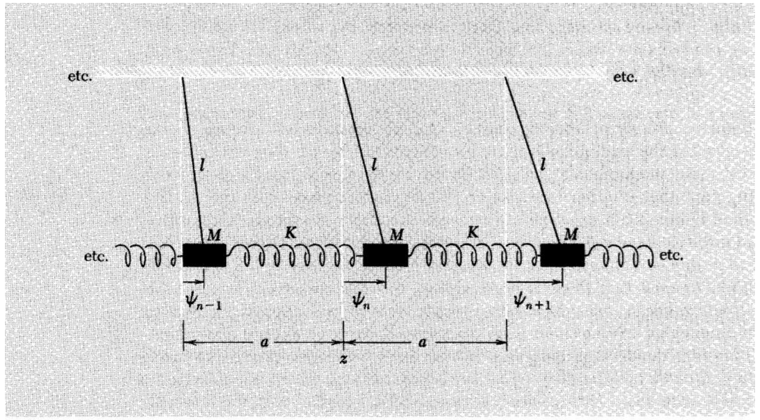


Fig. 3.10 Coupled pendulums with unspecified boundary conditions.

$$\boxed{\frac{\partial^2\psi(z,t)}{\partial t^2} = -\omega_0{}^2\psi(z,t) + \frac{Ka^2}{M}\frac{\partial^2\psi(z,t)}{\partial z^2}.} \tag{63}$$

Klein-Gordon wave equation. Equation (63) is a famous wave equation. It is not the classical wave equation, except when ω_0 is zero. It is sometimes called the "Klein-Gordon wave equation." (It holds for the de Broglie waves of relativistic free particles. See Supplementary Topic 2.)

Esoteric examples

If one combines De Broglie's hypothesis, which says that a particle of momentum p has a wave number k given by $p = \hbar k$, with the "Bohr frequency condition," which says that a particle of energy E has a wave frequency ω given by $E = \hbar\omega$, one can then find a dispersion relation between ω and k for particles, given the relation between E and p. Examples are given in Supplementary Topic 2.

For a relativistic free particle, the relation between energy, momentum, and rest mass m is given by

$$E^2 = (mc^2)^2 + (cp)^2, \tag{5}$$

which gives the dispersion relation (using $E = \hbar\omega$ and $p = \hbar k$, which are relativistically correct)

$$\hbar^2\omega^2 = (mc^2)^2 + (\hbar ck)^2. \tag{6}$$

Eq. (7). For free relativistic particles, the relativistic dispersion relation is

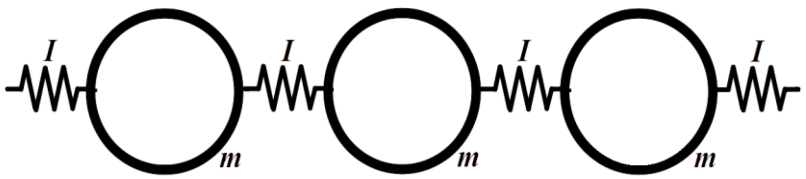
$$\hbar^2\omega^2 = \hbar^2c^2k^2 + (mc^2)^2. \tag{8}$$

Multiplying Eq. (8) by $-\hbar^{-2}\psi(z,t)$ and using Eqs. (3) and (5), we obtain

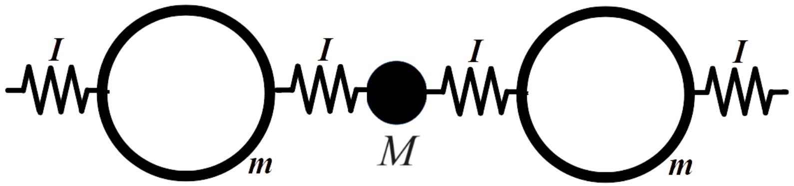
$$\frac{\partial^2\psi(z,t)}{\partial t^2} = c^2\frac{\partial^2\psi(z,t)}{\partial z^2} - \frac{(mc^2)}{\hbar^2}\psi(z,t). \tag{9}$$

Equation (9) is called the *Klein-Gordon equation.* Notice that if we set

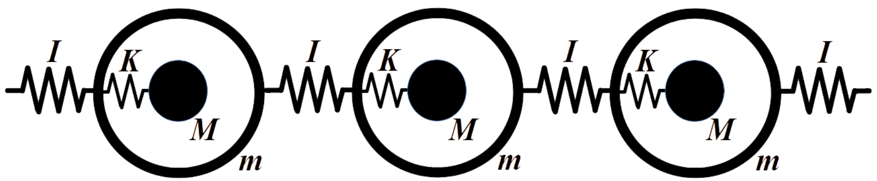
ONE-DIMENSIONAL MASS-IN-MASS CHAINS



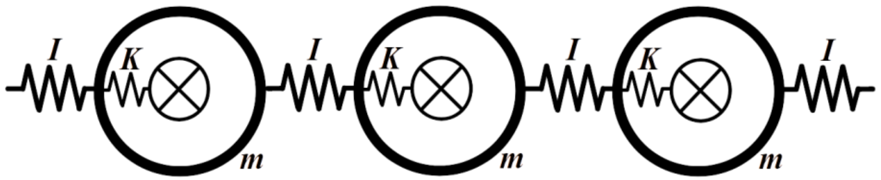
A one-dimensional chain with identical masses. A model of a one-dimensional monatomic crystal with only an acoustic dispersion branch



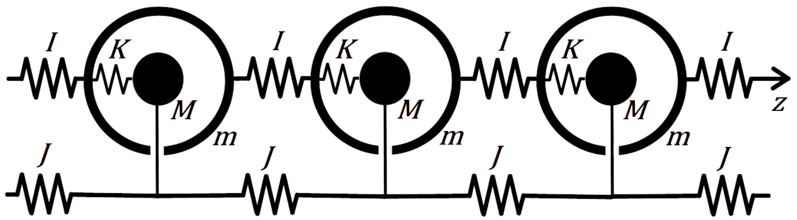
A one-dimensional chain with two different masses. A model of a one-dimensional diatomic crystal with acoustic and optical dispersion branches



One-dimensional chain "mass in mass“. J. H. Vincent 1989

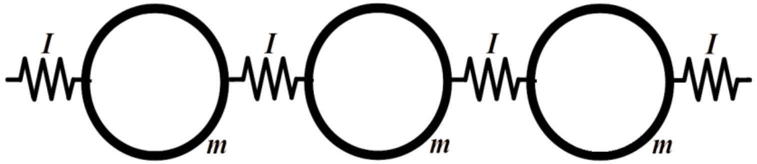


A chain of connected oscillators is a mass-in-mass chain in the case of M ≫ m (the equilibrium position of the load m is fixed and marked with a cross)



A modified mass-in-mass chain with the addition of harmonic interaction between loads with the same mass M

A ONE-DIMENSIONAL CHAIN WITH IDENTICAL MASSES.
A MODEL OF A ONE-DIMENSIONAL MONATOMIC CRYSTAL

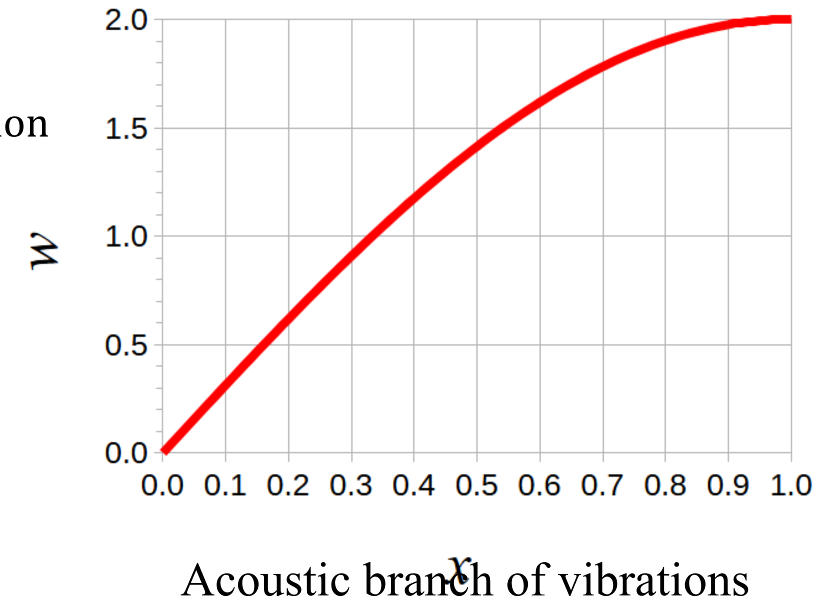


The equation of motion
(Newton's Second Law of Motion):

$$m\frac{d^2u_n}{dt^2} = I(u_{n-1} + u_{n+1} - 2u_n)$$

Only "acoustic" dispersion branch:

$$\omega^2 = \frac{4I}{m}\sin^2\frac{ka}{2}$$



Acoustic branch of vibrations

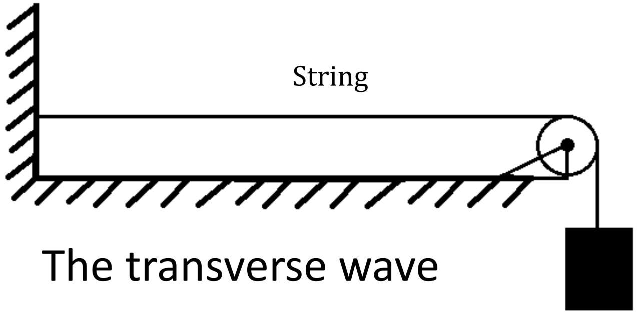
$$\omega_m = \sqrt{\frac{I}{m}} \qquad k = \frac{2\pi}{\lambda} \qquad \omega = \frac{2\pi}{T} \qquad w = \frac{\omega}{\omega_m} \qquad x = \frac{ka}{\pi}$$

$$w = 2\sin\frac{\pi x}{2}$$

In the case of the long-wave approximation (transition to a continuous string), the mass chain m is described by a wave equation with an only "acoustic" dispersion branch:

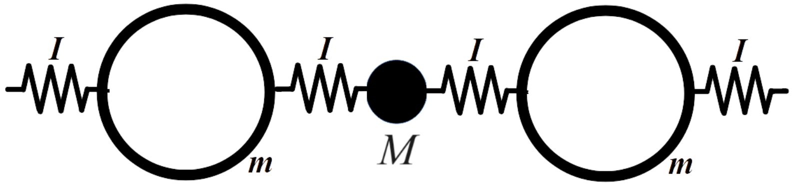
$$\frac{\partial^2 u}{\partial t^2} = s_m^2\frac{\partial^2 u}{\partial z^2}, \qquad \omega = s_m k, \qquad s_m = a\sqrt{\frac{I}{m}} = a\omega_m$$

$s_m = a\omega_m = \sqrt{E/\rho} = \sqrt{aI/\rho} = a\sqrt{I/m}$ - the phase velocity of the wave,
$E = aI$ - Young's module, $\rho = m/a$ - linear density.



The transverse wave

A ONE-DIMENSIONAL CHAIN WITH TWO DIFFERENT MASSES. A MODEL OF A ONE-DIMENSIONAL DIATOMIC CRYSTAL



$$\omega_{\pm}(k) = \omega_0^* \sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\mu}{m+M}\sin^2\frac{ka}{2}}}$$

$$\omega_0^* = \sqrt{\frac{2I}{\mu}} = \sqrt{2(1+\varepsilon)\frac{I}{m}} = \sqrt{2(1+\varepsilon)}\,\omega_m$$

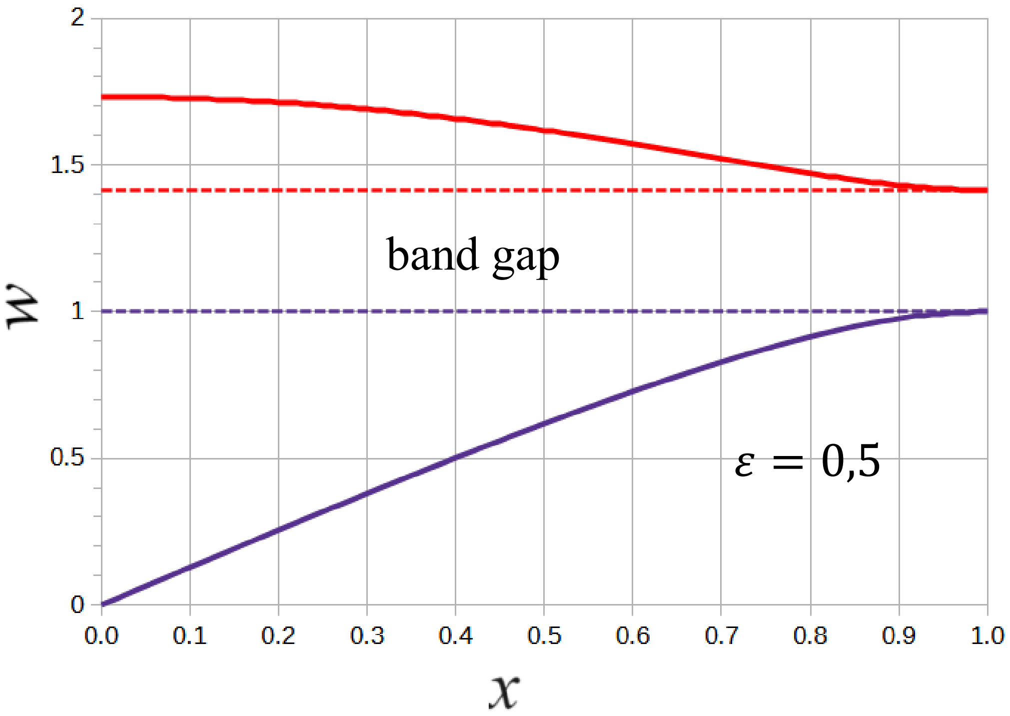
$$w_{\pm} = \sqrt{1+\varepsilon}\sqrt{1 \pm \sqrt{1 - \frac{4\varepsilon}{(1+\varepsilon)^2}\sin^2\frac{\pi x}{2}}}$$

$$w_{\pm} = \frac{\omega_{\pm}}{\omega_m} \qquad \varepsilon = \frac{m}{M} \qquad x = \frac{ka}{\pi} \qquad \omega_m = \sqrt{\frac{I}{m}}$$

$$\mu = \frac{mM}{m+M} = \frac{m}{\varepsilon+1}$$

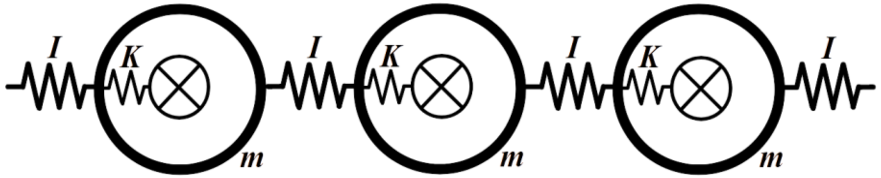
Here μ is the reduced atomic mass of a primitive cell.

There are acoustic ω_-, optical ω_+ dispersion branches and a band gap



The law of dispersion for a chain with two different masses. The optical mode ω_+ is a solid red line, and the acoustic mode ω_- is a blue dotted line. $\varepsilon = 0{,}5$

A CHAIN OF COUPLED OSCILLATORS (THE EQUILIBRIUM POSITION OF THE MASS M IS FIXED AND MARKED WITH A CROSS)



$$m\frac{d^2u_n}{dt^2} = -Ku_n + I(u_{n-1} + u_{n+1} - 2u_n)$$

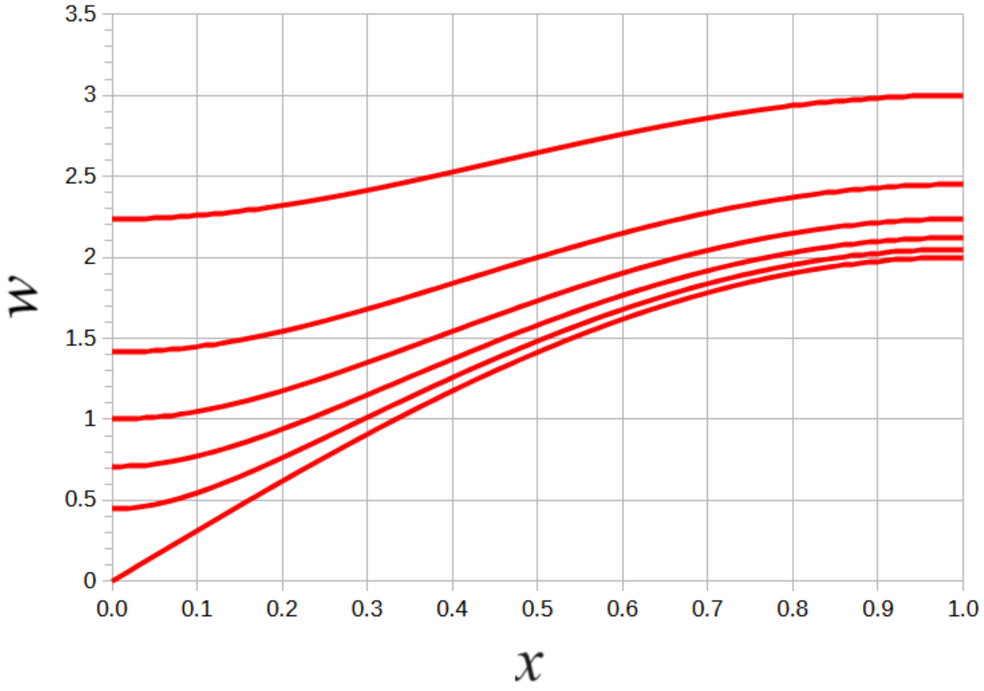
$$\omega^2 = \omega_{01}^2 + 4\omega_m^2 \sin^2\frac{ka}{2} \qquad \omega_m = \sqrt{\frac{I}{m}} \qquad \omega_{o1} = \sqrt{\frac{K}{m}}$$

$$y = \frac{I}{K} \qquad w_0 = \frac{\omega_{o1}}{\omega_m} = \frac{1}{\sqrt{y}} \qquad w = \frac{\omega}{\omega_m} \qquad x = \frac{ka}{\pi}$$

$$w = \sqrt{w_0^2 + 4\sin^2\frac{\pi x}{2}}$$

In the limit of long-wave oscillations, the chain is described by the real Klein-Fock-Gordon equation (classical physics)

$$\frac{\partial^2 u}{\partial t^2} = s_m^2\frac{\partial^2 u}{\partial z^2} - \omega_{01}^2 u$$



The law of dispersion. The parameter K/I = 1/y takes the value from 0 on the lower curve, then 0.2; 0.5; 1; 2 and 5 on the upper one, respectively, the parameter y takes the value ∞ on the lower curve, then 5; 2; 1; 0.5; 0.2 on the upper one. The lower curve (K<<I) coincides with the graph for a one-dimensional infinite chain with the same masses m connected by springs of rigidity K.

Frank S. Crawford. Waves. Berkeley Physics Course. Vol. 3. New York: McGraw-Hill, 1968

A CHAIN OF COUPLED OSCILLATORS.
LONG-WAVE APPROXIMATION

In the case of the long-wave approximation (transition to a continuous string), the chain of coupled oscillators is described by the real Klein-Gordon-Fock equation (KGF) with an "optical" dispersion branch:

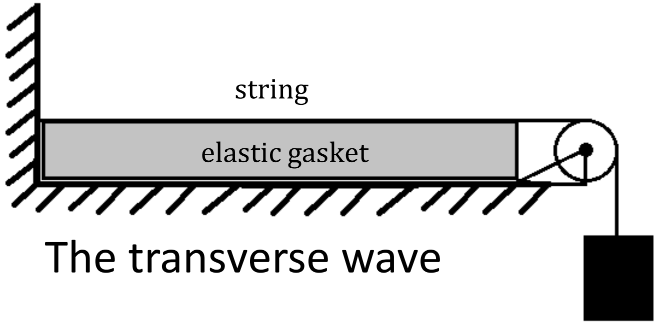
The real equation of KGF: $$\frac{\partial^2 u}{\partial t^2} = s_m^2 \frac{\partial^2 u}{\partial z^2} - \omega_{01}^2 u$$

$$\omega^2 = \omega_{01}^2 + s_m^2 k^2$$

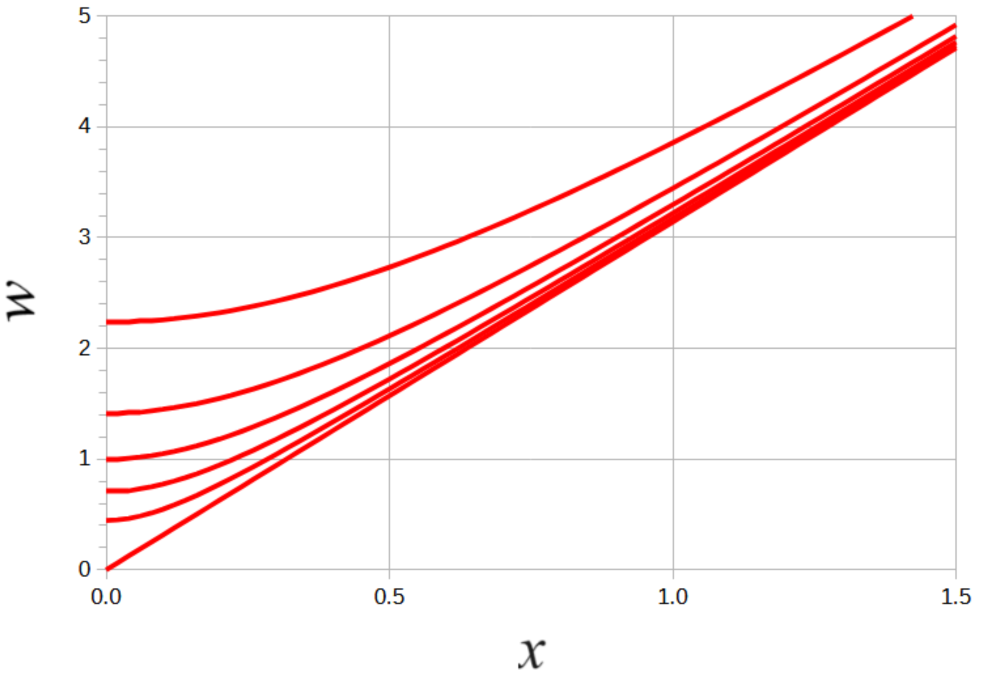
$$\omega_{01} = \sqrt{\frac{K}{m}}, \qquad s_m = a\omega_m , \qquad \omega_m = \sqrt{\frac{I}{m}}$$

$$w^2 = w_0^2 + \pi^2 x^2$$

Resembles: $$E^2 = (mc^2)^2 + p^2c^2$$

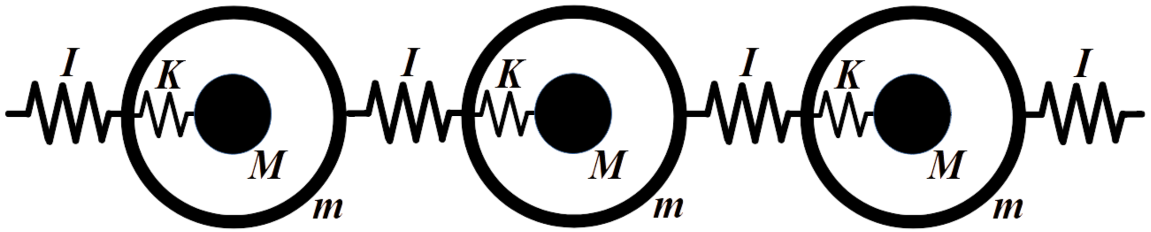


The transverse wave



The law of dispersion. The parameter K/I = 1/y takes the value from 0 on the lower curve, then 0.2; 0.5; 1; 2 and 5 on the upper one), respectively, the parameter *y* takes the value ∞ on the lower curve, then 5; 2; 1; 0.5; 0.2 on the upper one. The lower curve (K<<I) coincides with the law of dispersion for the wave equation $\omega_{01} = 0$.

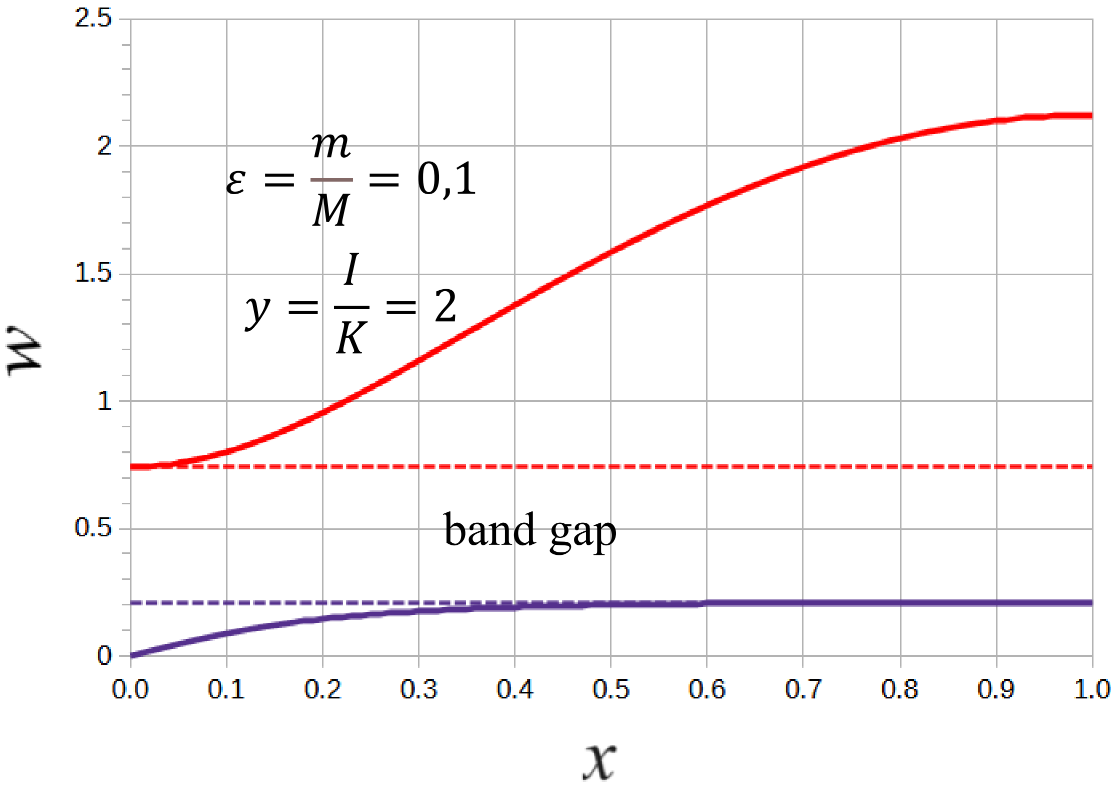
VINCENT'S CLASSIC MASS-IN-MASS CHAIN. 1898



$$\begin{cases} m\dfrac{d^2u_n}{dt^2} = K(U_n - u_n) + I(u_{n-1} + u_{n+1} - 2u_n) \\ M\dfrac{d^2U_n}{dt^2} = K(u_n - U_n) \end{cases}.$$

$$w_{\pm}^2 = \frac{1+\varepsilon}{2y}\left(1 + \frac{4y}{1+\varepsilon}\sin^2\frac{\pi x}{2}\right) \pm$$
$$\pm\frac{1+\varepsilon}{2y}\sqrt{\left(1 + \frac{4y}{1+\varepsilon}\sin^2\frac{\pi x}{2}\right)^2 - \frac{16\varepsilon y}{(1+\varepsilon)^2}\sin^2\frac{\pi x}{2}}$$

$$w_{\pm} = \frac{\omega_{\pm}}{\omega_m} \qquad \varepsilon = \frac{m}{M} \qquad y = \frac{I}{K} \qquad \omega_m = \sqrt{\frac{I}{m}}$$



There are acoustic ω_-, optical ω_+ dispersion branches and a band gap

The mass-in-mass chain is interesting from the point of view of creating acoustic metamaterials with unique characteristics. Acoustic metamaterials have developed from the research and findings in photonic (or optical) metamaterials. A novel optical metamaterial was originally proposed by Victor Veselago in 1967, but not realized until some 33 years later. John Pendry produced the basic elements of metamaterials in the late 1990s. His materials were combined, with negative index materials first realized in 2000, broadening the possible optical and material responses. Research in acoustic metamaterials has the same goal of broader material responses with sound waves.

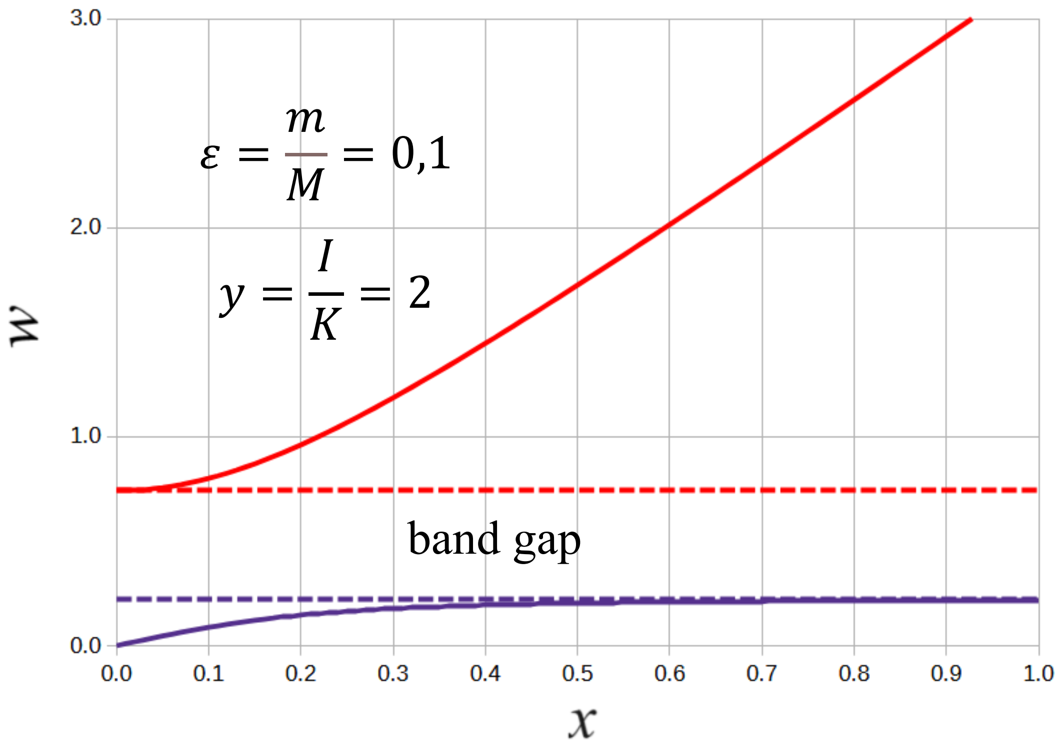
VINCENT'S CLASSIC MASS-IN-MASS CHAIN. LONG-WAVE APPROXIMATION

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = s_m^2 \dfrac{\partial^2 u}{\partial z^2} - \omega_{01}^2 (u - U) \\ \dfrac{\partial^2 U}{\partial t^2} = -\omega_{02}^2 (U - u) \end{cases}$$

$$w_\pm = \sqrt{\frac{1+\varepsilon}{2y}} \sqrt{1 + \frac{y}{1+\varepsilon}\pi^2 x^2 \pm \sqrt{\left(1 + \frac{y}{1+\varepsilon}\pi^2 x^2\right)^2 - \frac{4\varepsilon y}{(1+\varepsilon)^2}\pi^2 x^2}}$$

$$w_\pm = \frac{\omega_\pm}{\omega_m}$$

$$\omega_{01} = \sqrt{\frac{K}{m}} \quad \omega_{02} = \sqrt{\frac{K}{M}} \quad \omega_m = \sqrt{\frac{I}{m}} \quad s_m = a\omega_m$$

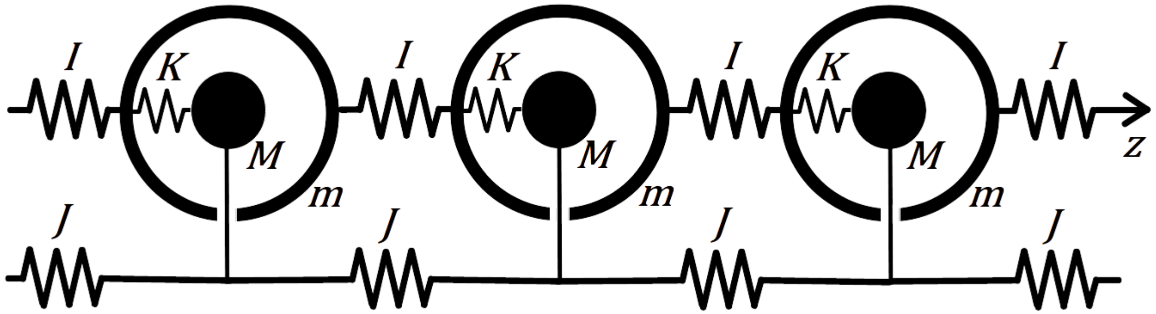


There are acoustic ω_-, optical ω_+ dispersion branches and a band gap

MODIFIED MASS-IN-MASS CHAIN

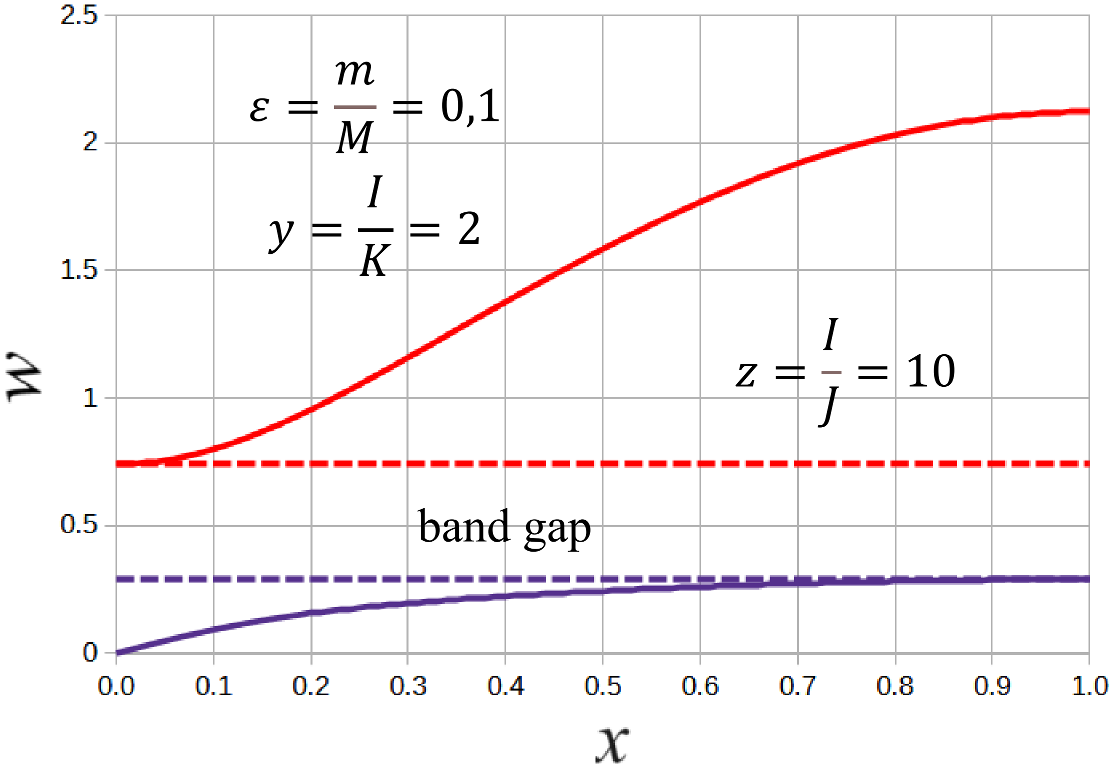
$$\begin{cases} m\dfrac{d^2u_n}{dt^2} = K(U_n - u_n) + I(u_{n-1} + u_{n+1} - 2u_n) \\ M\dfrac{d^2U_n}{dt^2} = K(u_n - U_n) + J(U_{n-1} + U_{n+1} - 2U_n) \end{cases}.$$

$$w_\pm^2 = \frac{1+\varepsilon}{2y}\left(1 + \left(1 + \frac{\varepsilon}{z}\right)\frac{4y}{1+\varepsilon}\sin^2\frac{\pi x}{2}\right) \pm$$
$$\pm \frac{1+\varepsilon}{2y}\sqrt{\left(1 + \left(1+\frac{\varepsilon}{z}\right)\frac{4y}{1+\varepsilon}\sin^2\frac{\pi x}{2}\right)^2 - \left(1 + \frac{1}{z}\left(1 + 4y\sin^2\frac{\pi x}{2}\right)\right)\frac{16\varepsilon y}{(1+\varepsilon)^2}\sin^2\frac{\pi x}{2}}$$



$$\omega_m = \sqrt{\frac{I}{m}} \qquad \omega_M = \sqrt{\frac{J}{M}}$$

$$\varepsilon = \frac{m}{M} \qquad y = \frac{I}{K} \qquad z = \frac{I}{J} \qquad w_\pm = \frac{\omega_\pm}{\omega_m}$$



Турин В.О., Назрицкий И.В., Киреев Д.Д., Андреев П.А., Илюшина Ю.В. Модифицированная цепочка масса-в-массе. Известия высших учебных заведений. Материалы электронной техники. 2024;27(4):330-340.

There are acoustic and optical dispersion branches. There may or may not be a band gap

MODIFIED MASS-IN-MASS CHAIN. LONG-WAVE APPROXIMATION. GENERALIZATION OF THE REAL KGF EQUATION

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = s_m^2 \dfrac{\partial^2 u}{\partial z^2} - \omega_{01}^2 (u - U) \\ \dfrac{\partial^2 U}{\partial t^2} = s_M^2 \dfrac{\partial^2 U}{\partial z^2} - \omega_{02}^2 (U - u) \end{cases}$$

$$\omega_{01} = \sqrt{\frac{K}{m}} \quad \omega_{02} = \sqrt{\frac{K}{M}} \quad \omega_m = \sqrt{\frac{I}{m}} \quad \omega_M = \sqrt{\frac{J}{M}}$$

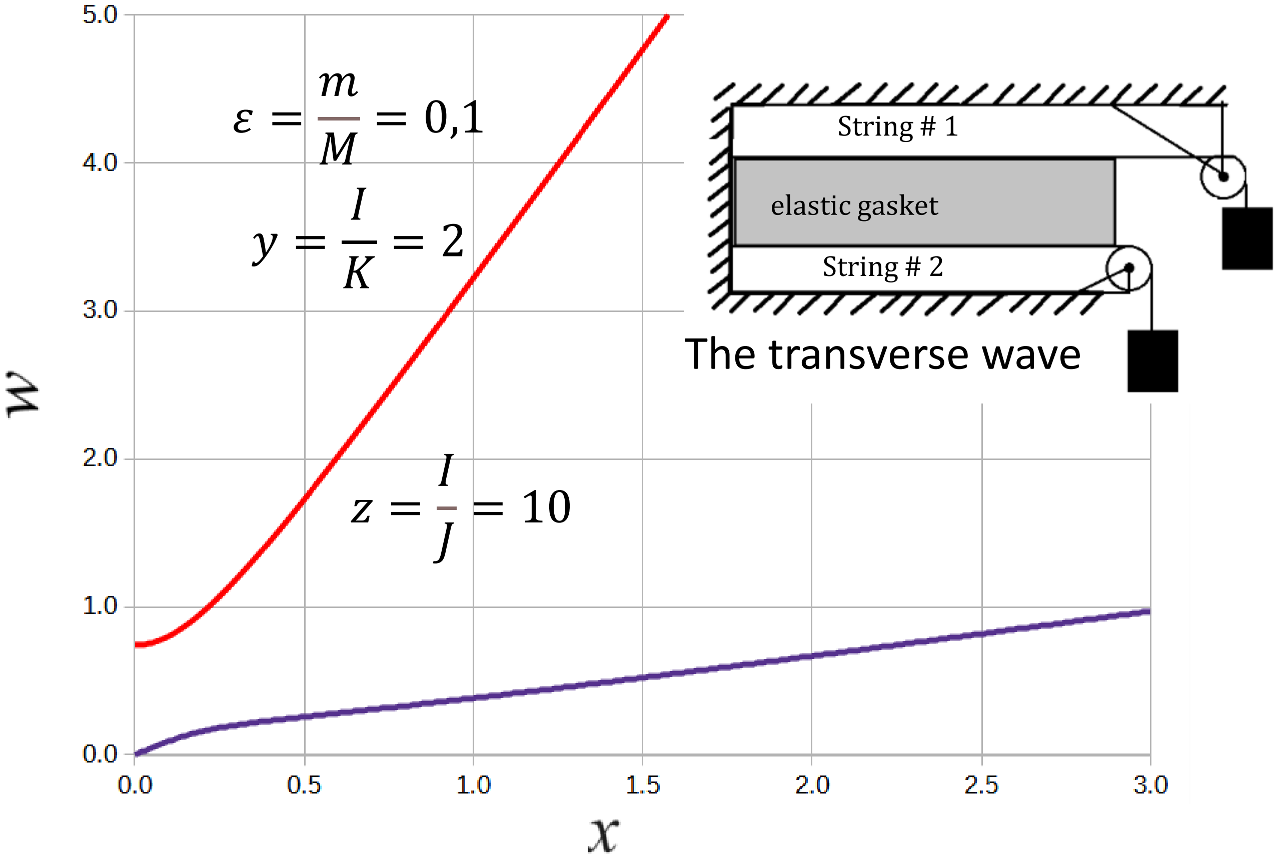
$$s_m = a\omega_m \quad s_M = a\omega_M \quad \varepsilon = \frac{m}{M} \quad y = \frac{I}{K} \quad z = \frac{I}{J}$$

Special case:

$\omega_m = \omega_M$ => $s_m = s_M = s$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = s^2 \dfrac{\partial^2 u}{\partial z^2} - \omega_{01}^2 (u - U) \\ \dfrac{\partial^2 U}{\partial t^2} = s^2 \dfrac{\partial^2 U}{\partial z^2} - \omega_{02}^2 (U - u) \end{cases}$$

$$w_\pm^2 = \frac{1+\varepsilon}{2y} + \frac{\left(1+\frac{\varepsilon}{z}\right)\pi^2 x^2}{2} \pm \sqrt{\left(\frac{1+\varepsilon}{2y} + \frac{\left(1+\frac{\varepsilon}{z}\right)\pi^2 x^2}{2}\right)^2 - \frac{\varepsilon}{y}\left(1 + \frac{1}{z}\left(1 + y\pi^2 x^2\right)\right)\pi^2 x^2}$$



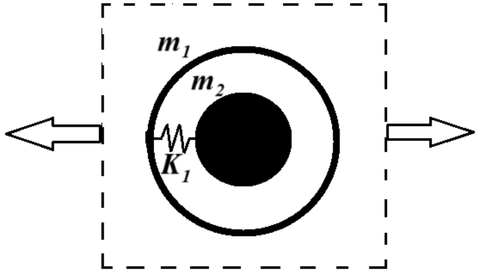
The transverse wave

There are acoustic ω_- and optical ω_+ dispersion branches.

THE CASE OF EQUALITY OF CHARACTERISTIC FREQUENCIES ω_m AND ω_M

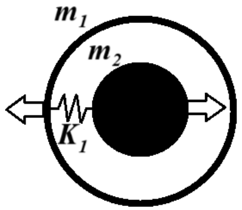
$$\omega_m = \sqrt{\frac{I}{m}} = \omega_M = \sqrt{\frac{J}{M}} \Rightarrow s_m = s_M = s$$

IN THIS CASE, THERE ARE ONLY TWO TYPES OF OSCILLATIONS



IN-PHASE OSCILLATION OF TWO MASSES

$$\omega_-^2 = 4\omega_m^2 \sin^2 \frac{ka}{2}$$



OSCILLATION OF TWO MASSES IN ANTIPHASE

$$\omega_+^2 = \omega_{O1}^2 + \omega_{O2}^2 + 4\omega_m^2 \sin^2 \frac{ka}{2}$$

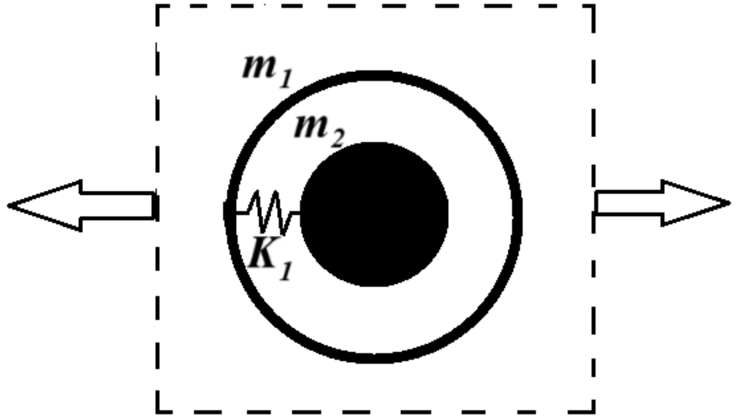
In the case of the long-wave approximation, for the acoustic branch of the dispersion, we have an equation that coincides with the only "acoustic" branch of the dispersion for the classical chain of identical masses in the case of the long-wave approximation.

$$\omega_-^2 = s^2 k^2 \qquad \omega_- = s\,k$$ - linear law of dispersion

For the optical branch of the dispersion, we have $\omega_+^2 = \omega_{O1}^2 + \omega_{O2}^2 + s^2 k^2$

The law of dispersion coincides in appearance with the single "optical" branch of the dispersion of the classical chain of connected identical oscillators in the case of the long-wave approximation and differs from it by adding a constant term to the right-hand side ω_{O2}^2.

IN-PHASE OSCILLATION OF TWO MASSES



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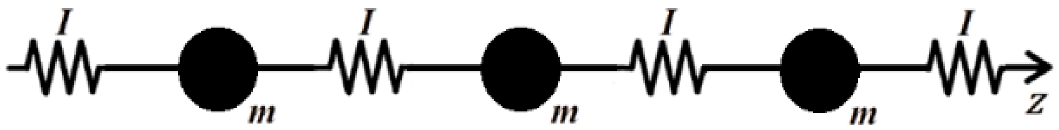
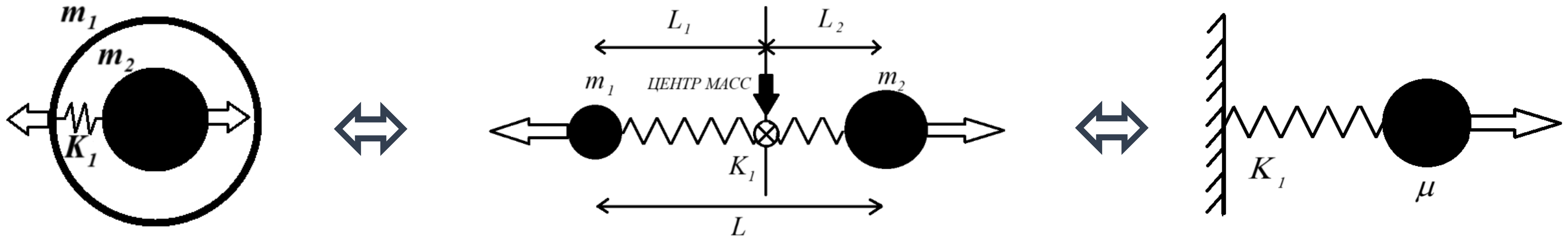


A chain of identical masses

The acoustic branch of the oscillations corresponds to the case when the masses m_1 and m_2 are stationary relative to each other (they move in phase - as a whole). At the same time, the internal spring K_1 remains undeformed all the time, and in two classical chains $(m_1,\ I_1)$ and $(m_2,$ $I_2)$, common-mode acoustic waves of the same amplitude, phase, and frequency propagate with the law of dispersion, which coincides with the acoustic law of dispersion for each chain separately:

$$\omega_- = 2\omega_{ch} \sin \frac{ka}{2}$$

THE PROBLEM OF TWO BODIES INTERACTING ACCORDING TO THE HARMONIC LAW



In the problem of two bodies interacting by harmonic law, mass m_1 and m_2 relative to each other fluctuate in antiphase around the center of mass, with frequency ω_{rd} corresponding fluctuations given mass μ spring stiffness K_1 and length L.

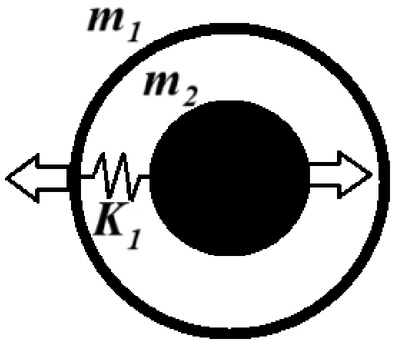
$$\mu = \frac{m_1 m_2}{m_1+m_2} \qquad \omega_{rd} = \sqrt{\frac{K_1}{\mu}}$$

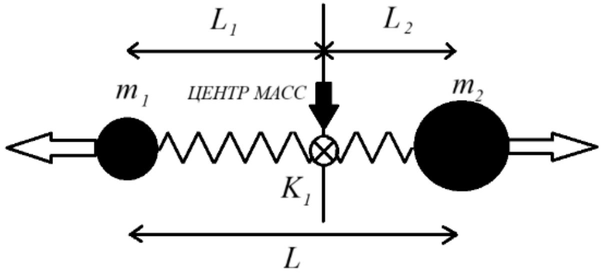
$$L_1 = \frac{m_2}{m_1+m_2} L \qquad L_2 = \frac{m_1}{m_1+m_2} L \qquad \frac{L_2}{L_1} = \frac{m_1}{m_2} \qquad L_1 + L_2 = L$$

$$K_{L1} = \frac{L}{L_1} K_1 = \frac{m_1+m_2}{m_2} K_1 = \frac{m_1}{\mu} K_1 \qquad K_{L2} = \frac{L}{L_2} K_1 = \frac{m_1+m_2}{m_1} K_1 = \frac{m_2}{\mu} K_1$$

$$\omega_{rd} = \sqrt{\frac{K_{L1}}{m_1}} = \sqrt{\frac{K_{L2}}{m_2}} = \sqrt{\frac{m_1+m_2}{m_1 m_2} K_1} = \sqrt{\frac{K_1}{\mu}}$$

OSCILLATIONS OF TWO MASSES IN ANTIPHASE





A chain of coupled oscillators

Optical branch of the oscillations corresponds to the case when the masses m_1 and m_2 relative to each other fluctuate in antiphase around the center of mass, with frequency ω_{rd} corresponding fluctuations given mass μ spring stiffness K_1. In this case, the type of the final law of dispersion coincides with the law of dispersion for a chain of identical coupled oscillators.

.

$$\omega_{+} = \sqrt{4\omega_{ch}^2 \sin^2\frac{ka}{2} + \omega_{rd}^2}$$

$$\omega_{ch} = \omega_1 = \sqrt{\frac{I_1}{m_1}} = \omega_2 = \sqrt{\frac{I_2}{m_2}} \qquad \omega_{rd} = \sqrt{\frac{K_1}{\mu}}$$

THE QUANTUM RELATIVISTIC KLEIN-FOCK-GORDON EQUATION

The real partial differential equation of the second order KGF $$\frac{\partial^2 u}{\partial t^2} = s_m^2 \frac{\partial^2 u}{\partial z^2} - \omega_0^2 u$$

The law of dispersion $$\omega^2 = \omega_0^2 + s_m^2 k^2$$

Formally corresponds to the complex-valued quantum-relativistic KGF equation ($u \leftrightarrow \Psi$, $s_m \leftrightarrow c$, $\omega_0 \leftrightarrow m_e c^2/\hbar$):

$$\frac{\partial^2 u}{\partial t^2} = s_m^2 \frac{\partial^2 u}{\partial z^2} - \omega_0^2 u \quad \leftrightarrow \quad \frac{\partial^2 \mathrm{Re}\Psi}{\partial t^2} = c^2 \frac{\partial^2 \mathrm{Re}\Psi}{\partial x^2} - \left(\frac{m_e c^2}{\hbar}\right)^2 \mathrm{Re}\Psi \quad \text{and} \quad \frac{\partial^2 \mathrm{Im}\Psi}{\partial t^2} = c^2 \frac{\partial^2 \mathrm{Im}\Psi}{\partial x^2} - \left(\frac{m_e c^2}{\hbar}\right)^2 \mathrm{Im}\Psi$$

$$\Psi = \mathrm{Re}\Psi + i\,\mathrm{Im}\Psi$$

$$\frac{\partial^2 u}{\partial t^2} = s_m^2 \frac{\partial^2 u}{\partial z^2} - \omega_0^2 u \quad \rightarrow \quad \frac{\partial^2 \Psi}{\partial t^2} = c^2 \frac{\partial^2 \Psi}{\partial x^2} - \left(\frac{m_e c^2}{\hbar}\right)^2 \Psi$$

THE QUANTUM-RELATIVISTIC DIRAC EQUATION

A further development of relativistic quantum theory is Dirac's system of first-order differential equations with a four-component wave function:

$$i\hbar \frac{\partial}{\partial t}|\Psi_4\rangle = \widehat{H}_{D_4}|\Psi_4\rangle \qquad \widehat{H}_{D_4} = \hbar\omega_O\alpha_0 + c\sum_{j=1}^{3}\alpha_j\,\hat{p}_j = m_e c^2\alpha_0 + c\sum_{j=1}^{3}\alpha_j\,\hat{p}_j$$

Here α_0, α_1, α_2, α_3 — Dirac alpha matrices4×4, expressed through a zero 2×2 matrix 0_2, unit 2×2 matrix I_2, and Pauli matrices 2×2

$$\alpha_0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix},\ \alpha_1 = \begin{pmatrix} 0_2 & \sigma_1 \\ \sigma_1 & 0_2 \end{pmatrix},\ \alpha_2 = \begin{pmatrix} 0_2 & \sigma_2 \\ \sigma_2 & 0_2 \end{pmatrix},\ \alpha_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ \sigma_3 & 0_2 \end{pmatrix},$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0_4,\quad i,j = 0,1,2,3\ (i \neq j) \quad \text{и} \quad \alpha_j^2 = I_4,\ \ j = 0,1,2,3.$$

Here I_4 – unit 4×4 matrix

From the Dirac equation, four independent KGF equations can be obtained, each for its own component of the wave function Ψ_i ($i = 1, 2, 3, 4$).

$$\frac{\partial^2\Psi_i}{\partial t^2} = c^2\frac{\partial^2\Psi_i}{\partial x^2} - \left(\frac{m_e c^2}{\hbar}\right)^2\Psi_i$$

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GENERALIZATION OF THE QUANTUM RELATIVISTIC KLEIN-FOCK-GORDON EQUATION

Based on a system for long-wave approximation for a modified mass-in-mass chain in case $s_m = s_M = s$

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = s_m^2 \frac{\partial^2 u}{\partial z^2} - \omega_{O1}^2 (u - U) \\ \frac{\partial^2 U}{\partial t^2} = s_M^2 \frac{\partial^2 U}{\partial z^2} - \omega_{O2}^2 (U - u) \end{cases} \rightarrow \begin{cases} \frac{\partial^2 u}{\partial t^2} = s^2 \frac{\partial^2 u}{\partial z^2} - \omega_{O1}^2 (u - U) \\ \frac{\partial^2 U}{\partial t^2} = s^2 \frac{\partial^2 U}{\partial z^2} - \omega_{O2}^2 (U - u) \end{cases}$$

using matching

$$u \leftrightarrow \Psi,\ U \leftrightarrow \Phi,\ s_m \leftrightarrow c,\ s_M \leftrightarrow c,\ \ \omega_{O1} = \sqrt{\frac{K}{m}} \leftrightarrow \frac{m_e c^2}{\hbar},\ \ \omega_{O2} = \sqrt{\frac{K}{M}} \leftrightarrow \frac{m_f c^2}{\hbar},\ \ \ m_f / m_e \leftrightarrow \sqrt{m/M}$$

We have constructed a system of second-order differential equations, which is a generalization of the quantum relativistic KGF equation $\frac{\partial^2 \Psi}{\partial t^2} = c^2 \frac{\partial^2 \Psi}{\partial x^2} - \left(\frac{m_e c^2}{\hbar}\right)^2 \Psi$:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = s^2 \frac{\partial^2 u}{\partial z^2} - \omega_{O1}^2 (u - U) \\ \frac{\partial^2 U}{\partial t^2} = s^2 \frac{\partial^2 U}{\partial z^2} - \omega_{O2}^2 (U - u) \end{cases} \rightarrow \begin{cases} \frac{\partial^2 \Psi}{\partial t^2} = c^2 \frac{\partial^2 \Psi}{\partial z^2} - \left(\frac{m_e c^2}{\hbar}\right)^2 (\Psi - \Phi) \\ \frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial z^2} - \left(\frac{m_f c^2}{\hbar}\right)^2 (\Phi - \Psi) \end{cases} \qquad \varepsilon = \frac{m}{M} = \left(\frac{m_f}{m_e}\right)^2$$

This system has "acoustic" and "optical" dispersion branches:

$$E^2 = c^2 p_z^2 \ \ \text{или} \ \ \Omega_A^2 = c^2 k_z^2 ,$$

$$E^2 = c^2 p_z^2 + (m_e c^2)^2 + \left(m_f c^2\right)^2 \ \text{or} \ \ \Omega_O^2 = c^2 k_z^2 + \left(\frac{m_e c^2}{\hbar}\right)^2 + \left(\frac{m_f c^2}{\hbar}\right)^2 .$$

Here $k_z = p_z / \hbar$ и $\Omega = E / \hbar$

GENERALIZATION OF THE QUANTUM-RELATIVISTIC DIRAC EQUATION

The following system of first-order differential equations is a generalization of the Dirac system of equations for a free electron in the one-dimensional case for the case of the projection of spin $\hbar/2$ on the axis z:

$$\begin{cases} i\hbar\dfrac{\partial\Psi_1}{\partial t} = -i\hbar c\dfrac{\partial\Psi_3}{\partial z} + \dfrac{m_e c^2}{\sqrt{1+\varepsilon^2}}(\Psi_1 - \Phi_1) \\ i\hbar\dfrac{\partial\Psi_3}{\partial t} = -i\hbar c\dfrac{\partial\Psi_1}{\partial z} - \dfrac{m_e c^2}{\sqrt{1+\varepsilon^2}}(\Psi_3 - \Phi_3) \\ i\hbar\dfrac{\partial\Phi_1}{\partial t} = -i\hbar c\dfrac{\partial\Phi_3}{\partial z} + \dfrac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}}(\Phi_1 - \Psi_1) \\ i\hbar\dfrac{\partial\Phi_3}{\partial t} = -i\hbar c\dfrac{\partial\Phi_1}{\partial z} - \dfrac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}}(\Phi_3 - \Psi_3) \end{cases}$$

Here $\varepsilon = \dfrac{m}{M} = \left(\dfrac{m_f}{m_e}\right)^2$.

$$\frac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}} = \frac{\varepsilon\, m_f c^2}{\sqrt{1+\varepsilon^2}} = \left(\frac{m_f}{m_e}\right)^3 \frac{m_e c^2}{\sqrt{1+\varepsilon^2}} = \varepsilon^{3/2}\,\frac{m_e c^2}{\sqrt{1+\varepsilon^2}}$$

The three-dimensional case:

$$\begin{cases} i\hbar\dfrac{\partial\Psi_1}{\partial t} = i\hbar c\left(-\dfrac{\partial\Psi_4}{\partial x} + i\dfrac{\partial\Psi_4}{\partial y} - \dfrac{\partial\Psi_3}{\partial z}\right) + \dfrac{m_e c^2}{\sqrt{1+\varepsilon^2}}(\Psi_1 - \Phi_1) \\ i\hbar\dfrac{\partial\Psi_2}{\partial t} = i\hbar c\left(-\dfrac{\partial\Psi_3}{\partial x} - i\dfrac{\partial\Psi_3}{\partial y} + \dfrac{\partial\Psi_4}{\partial z}\right) + \dfrac{m_e c^2}{\sqrt{1+\varepsilon^2}}(\Psi_2 - \Phi_2) \\ i\hbar\dfrac{\partial\Psi_3}{\partial t} = i\hbar c\left(-\dfrac{\partial\Psi_2}{\partial x} + i\dfrac{\partial\Psi_2}{\partial y} - \dfrac{\partial\Psi_1}{\partial z}\right) - \dfrac{m_e c^2}{\sqrt{1+\varepsilon^2}}(\Psi_3 - \Phi_3) \\ i\hbar\dfrac{\partial\Psi_4}{\partial t} = i\hbar c\left(-\dfrac{\partial\Psi_1}{\partial x} - i\dfrac{\partial\Psi_1}{\partial y} + \dfrac{\partial\Psi_2}{\partial z}\right) - \dfrac{m_e c^2}{\sqrt{1+\varepsilon^2}}(\Psi_4 - \Phi_4) \\ i\hbar\dfrac{\partial\Phi_1}{\partial t} = i\hbar c\left(-\dfrac{\partial\Phi_4}{\partial x} + i\dfrac{\partial\Phi_4}{\partial y} - \dfrac{\partial\Phi_3}{\partial z}\right) + \dfrac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}}(\Phi_1 - \Psi_1) \\ i\hbar\dfrac{\partial\Phi_2}{\partial t} = i\hbar c\left(-\dfrac{\partial\Phi_3}{\partial x} - i\dfrac{\partial\Phi_3}{\partial y} + \dfrac{\partial\Phi_4}{\partial z}\right) + \dfrac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}}(\Phi_2 - \Psi_3) \\ i\hbar\dfrac{\partial\Phi_3}{\partial t} = i\hbar c\left(-\dfrac{\partial\Phi_2}{\partial x} + i\dfrac{\partial\Phi_2}{\partial y} - \dfrac{\partial\Phi_1}{\partial z}\right) - \dfrac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}}(\Phi_3 - \Psi_3) \\ i\hbar\dfrac{\partial\Phi_4}{\partial t} = i\hbar c\left(-\dfrac{\partial\Phi_1}{\partial x} - i\dfrac{\partial\Phi_1}{\partial y} + \dfrac{\partial\Phi_2}{\partial z}\right) - \dfrac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}}(\Phi_4 - \Psi_4) \end{cases}$$

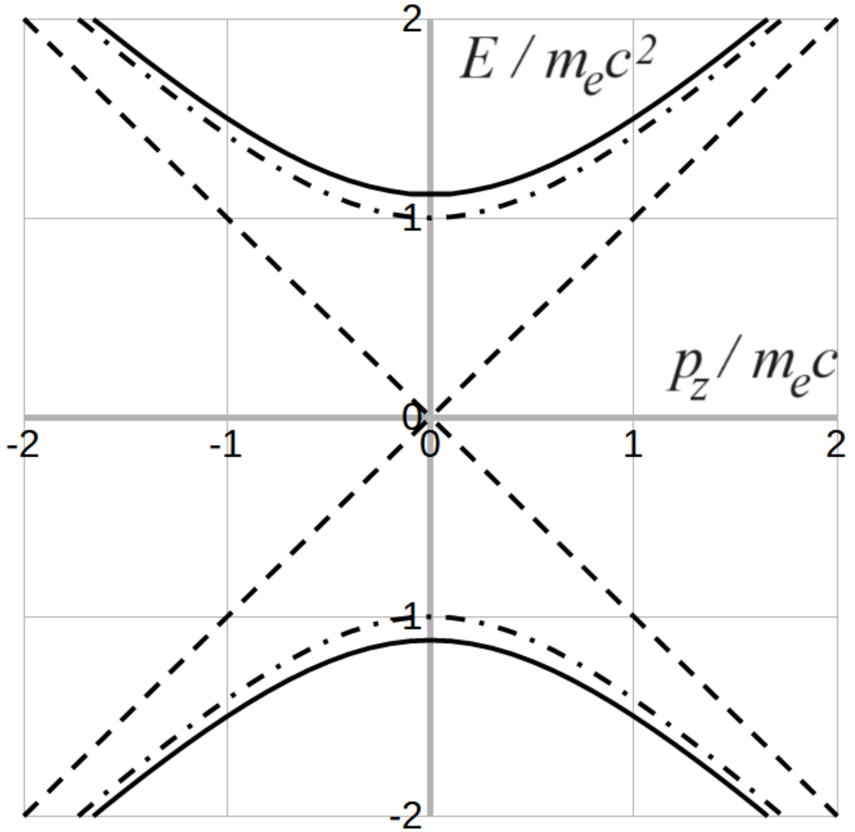
GENERALIZATION OF THE QUANTUM-RELATIVISTIC DIRAC EQUATION

This system has "acoustic" and "optical" dispersion branches:

$$E = \pm c\, p_z$$

$$E = \pm\sqrt{c^2 p_z^2 + (m_e c^2)^2 + \left(m_f c^2\right)^2} = \pm\sqrt{c^2\, p_z^2 + m_e^2 c^4 (1 + \varepsilon^2\,)}$$

All branches of the dispersion relation are presented. The dotted lines are for the linear case of variance. Solid lines for the nonlinear case of variance $E = \pm c\, p_z E = \sqrt{c^2\, p_z^2 + m_e^2 c^4 (1 + \varepsilon^2\,)}$ with the parameter $\varepsilon = 0.5$. Dotted lines for the nonlinear case of variance $E = \sqrt{c^2\, p_z^2 + m_e^2 c^4}$(parameters ε=0).



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GENERALIZED DIRAC EQUATION IN THE CASE OF THREE DIMENSIONS IN DIRAC'S NOTATION

$$i\hbar \frac{\partial}{\partial t}|\Psi_8\rangle = \widehat{H}_{D_8}|\Psi_8\rangle\,,\ \widehat{H}_{D_8} = \frac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}} A_{0-} + \frac{m_e c^2}{\sqrt{1+\varepsilon^2}} A_{0+} + c\ \sum_{j=1}^{3} A_j\, \hat{p}_j$$

Here $|\Psi_8\rangle$ is an eight-component complex wave function; A_{0-}, A_{0+}, A_1, A_2, A_3 - 4x4 matrices, which are a generalization of Dirac alpha matrices.

For generalized Dirac alpha matrices, the following relations hold:

$$A_{0-}^2 = \mathrm{A}_{0-}\mathrm{A}_{0+} = \begin{pmatrix} 0_4 & 0_4 \\ -I_4 & I_4 \end{pmatrix},\quad A_{0+}^2 = \mathrm{A}_{0+}\mathrm{A}_{0-} = \begin{pmatrix} I_4 & -I_4 \\ 0_4 & 0_4 \end{pmatrix},\ A_j^{\,2} = I_8,\ j = 1,2,3,$$

$$A_{0-} = \begin{pmatrix} 0_4 & 0_4 \\ -\alpha_0 & \alpha_0 \end{pmatrix}, A_{0+} = \begin{pmatrix} \alpha_0 & -\alpha_0 \\ 0_4 & 0_4 \end{pmatrix}, A_1 = \begin{pmatrix} \alpha_1 & 0_4 \\ 0_4 & \alpha_1 \end{pmatrix}, A_2 = \begin{pmatrix} \alpha_2 & 0_4 \\ 0_4 & \alpha_2 \end{pmatrix}, A_3 = \begin{pmatrix} \alpha_3 & 0_4 \\ 0_4 & \alpha_3 \end{pmatrix}$$

$$A_{0+}A_{0-} + A_{0-}A_{0+} = A_{0-}^2 + A_{0+}^2 = \begin{pmatrix} I_4 & -I_4 \\ -I_4 & I_4 \end{pmatrix}$$

$$\mathrm{A}_{0-}\mathrm{A}_j + \mathrm{A}_j\mathrm{A}_{0-} = 0_8,\ \ \mathrm{A}_{0+}\mathrm{A}_j + \mathrm{A}_j\mathrm{A}_{0+} = 0_8\,,\ \ j = 1,2,3,$$

$$\mathrm{A}_i\mathrm{A}_j + \mathrm{A}_j\mathrm{A}_i = 0_8\,, i,j = 1,2,3\ (i \neq j).$$

Here 0_8 and I_8 are the zero and unit matrices of dimension 8×8. The generalized Dirac equation yields four independent systems of generalized KGF equations, each for its own pair of wave function components Ψ_i and Φ_i ($i = 1,2,3,4$).

COVARIANCE OF THE GENERALIZED DIRAC EQUATION

In the stationary inertial frame of reference: $\hat{E} - \hat{H} = \hat{E} - \frac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}} A_{0-} - \frac{m_e c^2}{\sqrt{1+\varepsilon^2}} A_{0+} - c\,\hat{p}_x A_1 - c\,\hat{p}_y A_2 - c\,\hat{p}_z A_3$

Here $\hat{E} = i\hbar \frac{\partial}{\partial t}$, $\hat{p}_x = -i\hbar c \frac{\partial}{\partial x}$, $\hat{p}_y = -i\hbar c \frac{\partial}{\partial y}$, $\hat{p}_z = -i\hbar c \frac{\partial}{\partial z}$

We express $\hat{E} - \hat{H}$ in the stationary inertial frame of reference using $\hat{E}'$ and $\hat{p}'_x, \hat{p}'_y, \hat{p}'_z$ in moving inertial frame of reference

(with velocity *v* along the axis *x*) :

$$\hat{E} - \hat{H} = \gamma\left(\hat{E}' + v\hat{p}'_x\right) - \frac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}} A_{0-} - \frac{m_e c^2}{\sqrt{1+\varepsilon^2}} A_{0+} - c\gamma\left(\hat{p}'_x + \frac{v}{c^2}\hat{E}'\right) A_1 - c\,A_2 \hat{p}'_y - c\,A_3\,\hat{p}'_z = \cdots$$

$$= \left(\hat{E}' - c\hat{p}'_x A_1\right)(a + bA_1)\gamma\left(1 - \frac{v}{c}A_1\right) - \left(\frac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}} A_{0-} + \frac{m_e c^2}{\sqrt{1+\varepsilon^2}} A_{0+} + cA_2\,\hat{p}'_y + cA_3\,\hat{p}'_z\right)(a - bA_1)$$

Here $\hat{E}' = i\hbar \frac{\partial}{\partial t'}$, $\hat{p}'_x = -i\hbar c \frac{\partial}{\partial x}$, $\hat{p}'_y = -i\hbar c \frac{\partial}{\partial y'}$, $\hat{p}'_z = -i\hbar c \frac{\partial}{\partial z'}$, $\gamma = \frac{1}{\sqrt{1+\left(\frac{v}{c}\right)^2}}$

If $(a + bA_1)\gamma\left(1 - \frac{v}{c}A_1\right) = a - bA_1$ we have $(a + bA_1)\left(\hat{E} - \hat{H}\right) = \left(\hat{E}' - \hat{H}'\right)(a - bA_1)$

$\left(\hat{E}' - \hat{H}'\right)\Psi' = 0$ there is a generalized Dirac equation in a moving inertial frame of reference. Covariance is proven!

$$\hat{E}' - \hat{H}' = \hat{E}' - \frac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}} A_{0-} - \frac{m_e c^2}{\sqrt{1+\varepsilon^2}} A_{0+} - c\hat{p}'_x A_1 - c\hat{p}'_y A_2 - c\hat{p}'_z A_3$$

$$\Psi' = (a - bA_1)\Psi$$

It can be shown that $a = \sqrt{\frac{\gamma+1}{2}}$ и $b = \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}$. Accordingly, we find an explicit expression for Ψ':

$$\Psi' = \begin{pmatrix} \Psi'_1 \\ \Psi'_2 \\ \Psi'_3 \\ \Psi'_4 \\ \Phi'_1 \\ \Phi'_2 \\ \Phi'_3 \\ \Phi'_4 \end{pmatrix} = \left(\sqrt{\frac{\gamma+1}{2}} I_8 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}} A_1\right)\Psi = \begin{pmatrix} \sqrt{\frac{\gamma+1}{2}}\Psi_1 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Psi_4 \\ \sqrt{\frac{\gamma+1}{2}}\Psi_2 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Psi_3 \\ \sqrt{\frac{\gamma+1}{2}}\Psi_3 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Psi_2 \\ \sqrt{\frac{\gamma+1}{2}}\Psi_4 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Psi_1 \\ \sqrt{\frac{\gamma+1}{2}}\Phi_1 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Phi_4 \\ \sqrt{\frac{\gamma+1}{2}}\Phi_2 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Phi_3 \\ \sqrt{\frac{\gamma+1}{2}}\Phi_3 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Phi_2 \\ \sqrt{\frac{\gamma+1}{2}}\Phi_4 - \frac{\beta}{|\beta|}\sqrt{\frac{\gamma-1}{2}}\Phi_1 \end{pmatrix}$$

Approach from: Соколов, А.А. Квантовая теория поля / А. А. Соколов, Д. Д. Иваненко. — Москва ;, Ленинград : Гос. изд-во техн.-теорет. лит., 1952. — 780 с.

THE ORBITAL ANGULAR MOMENTUM AND SPIN FOR THE DIRAC EQUATION

The orbital angular momentum operator: $\hat{L} = \hat{r} \times \hat{p} = -i\hbar\ \hat{r} \times \nabla$

This is a vector operator, so it can be represented as the sum of projection operators: $\hat{L} = \hat{L}_x\vec{e}_x + \hat{L}_y\vec{e}_y + \hat{L}_z\vec{e}_z$

The angular momentum projection operators are defined by:

$$\hat{L}_x = y\hat{p}_z - z\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad \hat{L}_y = z\hat{p}_x - x\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \quad \hat{L}_z = x\hat{p}_y - y\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Dirac Hamiltonian for a particle moving in a centrally symmetric field: $\hat{H} = m_e c^2\alpha_0 + c\sum_{j=1}^{3}\alpha_j\,\hat{p}_j + V(r)r)$

We'll find it $[\hat{H}\hat{L}_x] = -i\hbar c(\alpha_2\hat{p}_z - \alpha_3\hat{p}_y)$ Matrices are introduced:

$$\sigma_x = \begin{pmatrix}\sigma_1 & 0_2\\ 0_2 & 1\end{pmatrix},\ \sigma_y = \begin{pmatrix}\sigma_2 & 0_2\\ 0_2 & \sigma_2\end{pmatrix},\ \sigma_z = \begin{pmatrix}\sigma_3 & 0_2\\ 0_2 & \sigma_3\end{pmatrix}$$

Pauli matrices: $\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

We have: $[\hat{H}\sigma_x] = m_e c^2[\alpha_0\sigma_x] + c\hat{p}_x[\alpha_1\sigma_x] + c\hat{p}_y[\alpha_2\sigma_x] + c\hat{p}_z[\alpha_3\sigma_x]$

It can be shown that: $[\alpha_0\sigma_x] = [\alpha_1\sigma_x] = 0_4$ $\quad [\alpha_2\sigma_x] = -2i\alpha_3$ $\quad [\alpha_3\sigma_x] = 2i\alpha_2$

We have:

$$\left[\hat{H}\ \frac{\hbar}{2}\sigma_x\right] = i\hbar c(\alpha_2\hat{p}_z - \alpha_3\hat{p}_y)$$

$$\Rightarrow \quad \left[\hat{H}\left(\hat{L}_x + \frac{\hbar}{2}\sigma_x\right)\right] = 0$$

The spin is equal to 1/2

THE ORBITAL ANGULAR MOMENTUM AND SPIN FOR THE GENERALIZED DIRAC EQUATION

Generalized Dirac Hamiltonian for a particle moving in a centrally symmetric field:

$$\hat{H} = \frac{m_f c^2}{\sqrt{1+\varepsilon^{-2}}} A_{0-} + \frac{m_e c^2}{\sqrt{1+\varepsilon^2}} A_{0+} + c \sum_{j=1}^{3} A_j \, \hat{p}_j + V(r)$$

We have:

$$[\hat{H}\hat{L}_x] = \cdots = i\hbar c\left(A_3 \hat{p}_y - A_2 \hat{p}_z\right)$$

We introduce matrices $\Sigma_{1,2,3}$:

$$\Sigma_x = \begin{pmatrix} \sigma_x & 0_4 \\ 0_4 & \sigma_x \end{pmatrix} \quad \Sigma_x = \begin{pmatrix} \sigma_y & 0_4 \\ 0_4 & \sigma_y \end{pmatrix} \quad \Sigma_x = \begin{pmatrix} \sigma_z & 0_4 \\ 0_4 & \sigma_z \end{pmatrix}$$

Матрицы $\sigma_{1,2,3}$ были введены при анализе уравнения Дирака:

$$\sigma_x = \begin{pmatrix} \sigma_1 & 0_2 \\ 0_2 & \sigma_1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} \sigma_2 & 0_2 \\ 0_2 & \sigma_2 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} \sigma_3 & 0_2 \\ 0_2 & \sigma_3 \end{pmatrix}$$

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have

$$[\hat{H}\Sigma_1] = \frac{m_f c^2}{\sqrt{1+\varepsilon^2}}[A_{0-}\,\Sigma_1] + \frac{m_e c^2}{\sqrt{1+\varepsilon^{-2}}}[A_{0+}\,\Sigma_1] + c\hat{p}_x[A_1\,\Sigma_1] + c\hat{p}_y[A_2\,\Sigma_1] + c\hat{p}_z[A_3\,\Sigma_1] + [V(r)\Sigma_1]$$

Можно показать, что:

$$[A_{0-}\,\Sigma_x] = [A_{0+}\,\Sigma_x] = 0_8 \qquad [A_2\,\Sigma_x] = -2iA_3 \qquad [A_3\,\Sigma_x] = 2iA_2$$

We have

$$\left[\hat{H}\frac{\hbar}{2}\Sigma_x\right] = -i\hbar c\left(A_3\hat{p}_y - A_2\hat{p}_z\right)$$

=> $\left[\hat{H}\left(\hat{L}_x + \frac{\hbar}{2}\Sigma_x\right)\right] = 0$ The spin is equal to 1/2

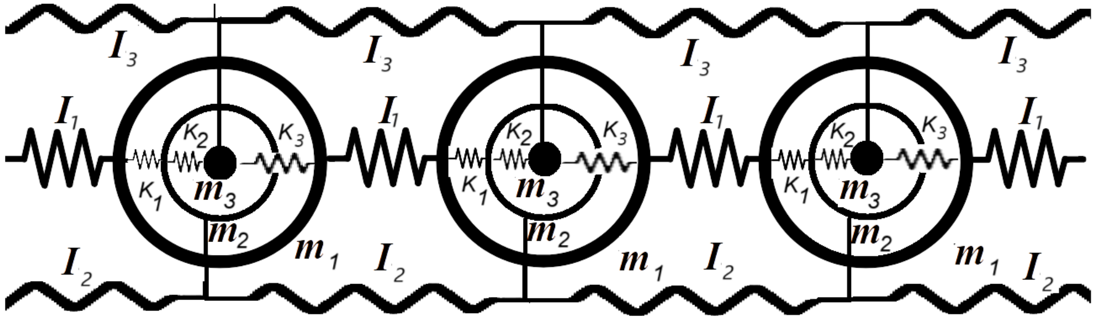
THE PROBLEM OF THE HIERARCHY OF FERMIONIC MASSES

The problem of the hierarchy of fermionic masses is one of the unsolved problems of particle physics and lies in the fact that the observed masses of the three generations of fermions (leptons and quarks) differ tenfold, despite the fact that the other properties of these particles and their quantum numbers are the same.

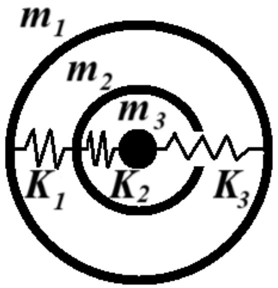
In the Standard Model, all fermions (both quarks and leptons) form three generations. Each generation is a collection of different types of particles, and the generations differ only in greatly varying masses. There are currently three generations of leptons. First generation: electron, electron neutrino. Second generation: muon, muon neutrino. Third generation: tau-lepton, tau-neutrino. An electron has a mass of 0.511 MeV, the mass of a muon is 105.7 MeV, and the mass of a tau lepton is already 1777 MeV. At the same time, all these particles have the same set of quantum numbers.

CHAIN MASS(-IN-MASS)^2

GENERALIZATION OF THE REAL KGF EQUATION



One-dimensional modified mass(-in-mass)^2 chain with the addition of a harmonic interaction between neighboring masses.



One cell of the modified chain mass(-in-mass)^2 is the problem of three bodies interacting according to a harmonic law

The equation of motion (Newton's Second Law of Motion):

$$\begin{cases} m_1 \dfrac{d^2 u_n}{dt^2} = K_1(v_n - u_n) + K_3(w_n - u_n) + I_1(u_{n-1} + u_{n+1} - 2u_n) \\ m_2 \dfrac{d^2 v_n}{dt^2} = K_2(w_n - v_n) + K_1(u_n - v_n) + I_2(v_{n-1} + v_{n+1} - 2v_n) \\ m_3 \dfrac{d^2 w_n}{dt^2} = K_3(u_n - w_n) + K_2(v_n - w_n) + I_3(w_{n-1} + w_{n+1} - 2w_n) \end{cases}$$

Generalization of the real KGF equation :

$$\begin{cases} \dfrac{d^2 u}{dt^2} = s_1^2 \dfrac{\partial^2 u}{\partial z^2} - \omega_{121}^2 (u - v) - \omega_{131}^2 (u - w) \\ \dfrac{d^2 v}{dt^2} = s_2^2 \dfrac{\partial^2 v}{\partial z^2} - \omega_{232}^2 (v - w) - \omega_{122}^2 (v - u) \\ \dfrac{d^2 w}{dt^2} = s_3^2 \dfrac{\partial^2 w}{\partial z^2} - \omega_{133}^2 (w - u) - \omega_{233}^2 (w - v) \end{cases}$$

$$\omega_{121} = \sqrt{\frac{K_1}{m_1}},\ \omega_{232} = \sqrt{\frac{K_2}{m_2}},\ \omega_{133} = \sqrt{\frac{K_3}{m_3}},\ \omega_{122} = \sqrt{\frac{K_1}{m_2}},\ \omega_{233} = \sqrt{\frac{K_2}{m_3}},\ \omega_{131} = \sqrt{\frac{K_3}{m_1}},\ \omega_1 = \sqrt{\frac{I_1}{m_1}},\ \omega_2 = \sqrt{\frac{I_2}{m_2}},\ \omega_3 = \sqrt{\frac{I_3}{m_3}}.$$

$$s_1 = a\omega_1, \qquad s_2 = a\omega_2, \qquad s_3 = a\omega_3$$

Generalization of the real KGF equation in case $\omega_1 = \omega_2 = \omega_3$:

$$\begin{cases} \frac{d^2u}{dt^2} = s^2\frac{\partial^2 u}{\partial z^2} - \omega_{121}^2(u-v) - \omega_{131}^2(u-w) \\ \frac{d^2v}{dt^2} = s^2\frac{\partial^2 v}{\partial z^2} - \omega_{232}^2(v-w) - \omega_{122}^2(v-u) \\ \frac{d^2w}{dt^2} = s^2\frac{\partial^2 w}{\partial z^2} - \omega_{133}^2(w-u) - \omega_{233}^2(w-v) \end{cases} \rightarrow \begin{cases} \frac{d^2\Psi}{dt^2} = c^2\frac{\partial^2 \Psi}{\partial z^2} - \left(\frac{m_e c^2}{\hbar}\right)^2(\Psi-\Phi) - \left(\frac{m_g c^2}{\hbar}\right)^2(\Psi-\Lambda) \\ \frac{d^2\Phi}{dt^2} = c^2\frac{\partial^2 \Phi}{\partial z^2} - \left(\frac{m_d c^2}{\hbar}\right)^2(\Phi-\Lambda) - \left(\frac{m_f c^2}{\hbar}\right)^2(\Phi-\Psi) \\ \frac{d^2\Lambda}{dt^2} = c^2\frac{\partial^2 \Lambda}{\partial z^2} - \left(\frac{m_m c^2}{\hbar}\right)^2(\Lambda-\Psi) - \left(\frac{m_h c^2}{\hbar}\right)^2(\Lambda-\Phi) \end{cases}$$

Like it was done for mass-in-mass chain:

$$\begin{cases} \frac{\partial^2u}{\partial t^2} = s^2\frac{\partial^2 u}{\partial z^2} - \omega_{O1}^2(u-U) \\ \frac{\partial^2U}{\partial t^2} = s^2\frac{\partial^2 U}{\partial z^2} - \omega_{O2}^2(U-u) \end{cases} \rightarrow \begin{cases} \frac{\partial^2\Psi}{\partial t^2} = c^2\frac{\partial^2 \Psi}{\partial z^2} - \left(\frac{m_e c^2}{\hbar}\right)^2(\Psi-\Phi) \\ \frac{\partial^2\Phi}{\partial t^2} = c^2\frac{\partial^2 \Phi}{\partial z^2} - \left(\frac{m_f c^2}{\hbar}\right)^2(\Phi-\Psi) \end{cases}$$

$$\varepsilon = \frac{m}{M} = \left(\frac{m_f}{m_e}\right)^2$$

CHAIN MASS(-IN-MASS)^2

We are looking for a solution to a system of equations in the form of waves with complex amplitudes.$\tilde{u}$, $\tilde{v}$ and $\tilde{w}$:

$$u_n = \tilde{u}e^{i(kz_n-\omega t)}\,, \quad v_n = \tilde{v}e^{i(kz_n-\omega t)}\,, \quad w_n = \tilde{w}e^{i(kz_n-\omega t)}.$$

After that, we obtain a system of homogeneous linear algebraic equations:

$$\begin{cases} \left(\omega^2 - 4\omega_1^2 \sin^2\frac{ka}{2} - \omega_{121}^2 - \omega_{131}^2\right)\tilde{u} + \omega_{121}^2\tilde{v} + \omega_{131}^2\tilde{w} = 0 \\ \omega_{122}^2\tilde{u} + \left(\omega^2 - 4\omega_2^2 \sin^2\frac{ka}{2} - \omega_{232}^2 - \omega_{122}^2\right)\tilde{v} + \omega_{232}^2\tilde{w} = 0 \\ \omega_{133}^2\tilde{u} + \omega_{233}^2\tilde{v} + \left(\omega^2 - 4\omega_3^2 \sin^2\frac{ka}{2} - \omega_{133}^2 - \omega_{233}^2\right)\tilde{w} = 0 \end{cases}.$$

Here

$$\omega_{121} = \sqrt{\frac{K_1}{m_1}}\,, \omega_{232} = \sqrt{\frac{K_2}{m_2}}\,, \omega_{133} = \sqrt{\frac{K_3}{m_3}}\,,$$

$$\omega_{122} = \sqrt{\frac{K_1}{m_2}}, \omega_{233} = \sqrt{\frac{K_2}{m_3}}\,, \omega_{131} = \sqrt{\frac{K_3}{m_1}}\,,$$

$$\omega_1 = \sqrt{\frac{I_1}{m_1}}\,, \omega_2 = \sqrt{\frac{I_2}{m_2}}\,, \omega_3 = \sqrt{\frac{I_3}{m_3}}\,.$$

CHAIN MASS(-IN-MASS)^2

Let us consider an interesting case of equality of the characteristic frequencies of all three chains.:

$$\omega_{I1} = \sqrt{\frac{I_1}{m_1}} = \omega_{I2} = \sqrt{\frac{I_2}{m_2}} = \omega_{I3} = \sqrt{\frac{I_3}{m_3}} = \omega_{ch}$$

The law of dispersion for a chain of identical masses is a classical one-dimensional infinite chain of masses m arranged with a period a and connected by springs with rigidity I :

$$\omega^2 = 4\omega_{ch}^2 \sin^2 \frac{ka}{2}$$

and

$$\omega_{1,2}^2 = 4\,\omega_{ch}^2 \sin^2 \frac{ka}{2} + \frac{|p| \pm \sqrt{p^2 - 4q}}{2}\,.$$

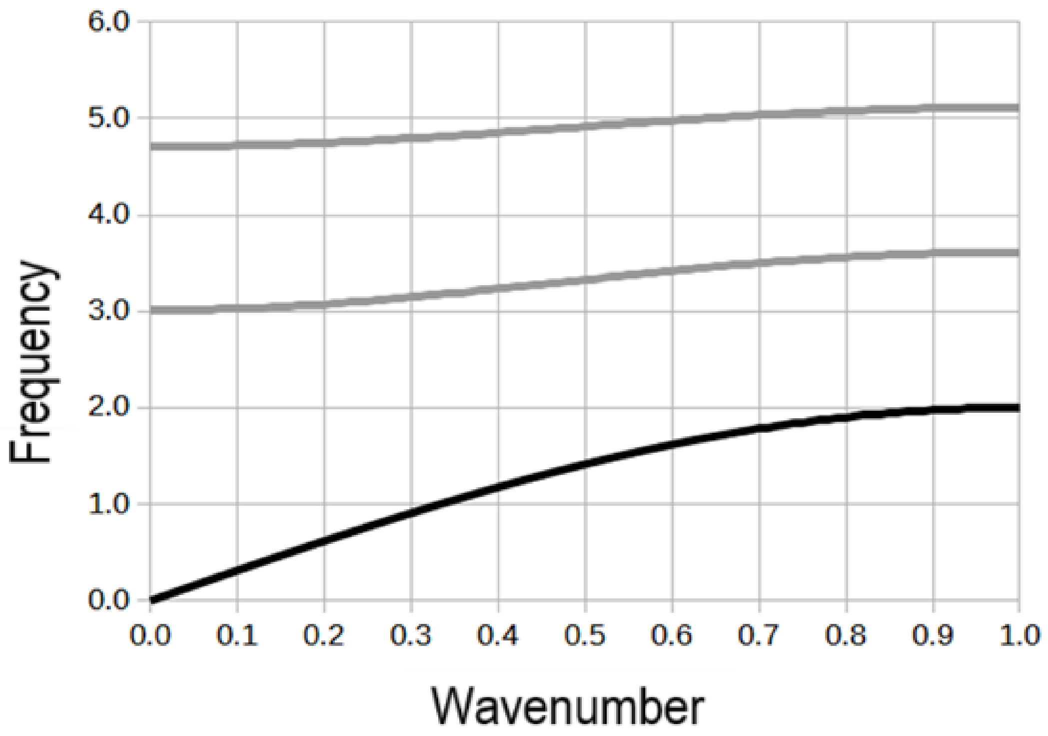
Here

$$|p| = \omega_{121}^2 + \omega_{131}^2 + \omega_{122}^2 + \omega_{232}^2 + \omega_{133}^2 + \omega_{233}^2$$

and

$$q = \omega_{122}^2\omega_{133}^2 + \omega_{122}^2\omega_{233}^2 + \omega_{133}^2\omega_{232}^2 + \\ +\omega_{121}^2\omega_{133}^2 + \omega_{121}^2\omega_{233}^2 + \omega_{131}^2\omega_{233}^2 + \\ +\omega_{121}^2\omega_{232}^2 + \omega_{122}^2\omega_{131}^2 + \omega_{131}^2\omega_{232}^2\,.$$

CHAIN MASS(-IN-MASS)^2



Acoustic (black lines) and optical (gray lines) dispersion branches for the modified mass-in-mass chain[2] at $\beta_1 = I_1/K_1 = 1$, $\varepsilon_{12}^2 = m_1/m_2$ =10, $\varepsilon_{23}^2 = m_2/m_3$ =2, $\kappa_{12} = K_1/K_2 = \kappa_{23} = K_2/K_3 = 2$. In this case $I_1/I_2 = \varepsilon_{12}^2 = 10$, $I_2/I_3 = \varepsilon_{23}^2 = 2$, $\beta_2 = I_2/K_2 = \beta_3 = I_3/K_3 = 0{,}2$. The frequency along the vertical axis is plotted in units of ω_{ch}, and the wavenumber along the horizontal axis is set in units of $k_{\max} = \pi/a$.

We consider modified one-dimensional infinite chains of mass(-in-mass)^2. Such chain can be considered as new classes of acoustic metamaterials. And there is the possibility of further generalization of the equations of relativistic quantum mechanics in the case of equality of the characteristic frequencies of individual chains of identical masses. It concerns the problem of the hierarchy of fermionic masses. In the chain mass (-in-mass)^2 two optical dispersion branches arise, which can correspond to two massive particles, within the framework of one mathematical model, which can correspond to two generations of fermions.

CHAIN MASS(-IN-MASS)^3

GENERALIZATION OF THE REAL KGF EQUATION

The equation of motion (Newton's Second Law of Motion):

$$\begin{cases} m_1 \dfrac{d^2u_n}{dt^2} = K_{12}(v_n - u_n) + K_{13}(w_n - u_n) + K_{14}(z_n - u_n) + I_1(u_{n-1} + u_{n+1} - 2u_n) \\ m_2 \dfrac{d^2v_n}{dt^2} = K_{23}(w_n - v_n) + K_{24}(z_n - v_n) + K_{12}(z_n - v_n) + I_2(v_{n-1} + v_{n+1} - 2v_n) \\ m_3 \dfrac{d^2w_n}{dt^2} = K_{34}(z_n - w_n) + K_{13}(u_n - w_n) + K_{23}(v_n - w_n) + I_3(w_{n-1} + w_{n+1} - 2w_n) \\ m_4 \dfrac{d^2z_n}{dt^2} = K_{14}(u_n - z_n) + K_{24}(v_n - z_n) + K_{34}(w_n - z_n) + I_3(w_{n-1} + w_{n+1} - 2w_n) \end{cases}$$

Generalization of the real KGF equation :

$$\begin{cases} \dfrac{d^2u}{dt^2} = s_1^2 \dfrac{\partial^2 u}{\partial z^2} - \omega_{121}^2(u - v) - \omega_{131}^2(u - w) - \omega_{141}^2(u - z) \\ \dfrac{d^2v}{dt^2} = s_2^2 \dfrac{\partial^2 v}{\partial z^2} - \omega_{232}^2(v - w) - \omega_{242}^2(v - z) - \omega_{122}^2(v - u) \\ \dfrac{d^2w}{dt^2} = s_3^2 \dfrac{\partial^2 w}{\partial z^2} - \omega_{343}^2(w - z) - \omega_{133}^2(w - u) - \omega_{233}^2(w - v) \\ \dfrac{d^2z}{dt^2} = s_4^2 \dfrac{\partial^2 w}{\partial z^2} - \omega_{144}^2(z - u) - \omega_{244}^2(z - v) - \omega_{344}^2(z - w) \end{cases}$$

Generalization of the real KGF equation in case $\omega_1 = \omega_2 = \omega_3 = \omega_4$:

$$\begin{cases} \dfrac{d^2u}{dt^2} = s^2 \dfrac{\partial^2 u}{\partial z^2} - \omega_{121}^2(u - v) - \omega_{131}^2(u - w) - \omega_{141}^2(u - z) \\ \dfrac{d^2v}{dt^2} = s^2 \dfrac{\partial^2 v}{\partial z^2} - \omega_{232}^2(v - w) - \omega_{242}^2(v - z) - \omega_{122}^2(v - u) \\ \dfrac{d^2w}{dt^2} = s^2 \dfrac{\partial^2 w}{\partial z^2} - \omega_{343}^2(w - z) - \omega_{133}^2(w - u) - \omega_{233}^2(w - v) \\ \dfrac{d^2z}{dt^2} = s^2 \dfrac{\partial^2 w}{\partial z^2} - \omega_{144}^2(z - u) - \omega_{244}^2(z - v) - \omega_{344}^2(z - w) \end{cases}$$

$$\begin{cases} \frac{d^2u}{dt^2} = s^2\frac{\partial^2 u}{\partial z^2} - \omega_{121}^2(u-v) - \omega_{131}^2(u-w) - \omega_{141}^2(u-z) \\ \frac{d^2v}{dt^2} = s^2\frac{\partial^2 v}{\partial z^2} - \omega_{232}^2(v-w) - \omega_{242}^2(v-z) - \omega_{122}^2(v-u) \\ \frac{d^2w}{dt^2} = s^2\frac{\partial^2 w}{\partial z^2} - \omega_{343}^2(w-z) - \omega_{133}^2(w-u) - \omega_{233}^2(w-v) \\ \frac{d^2z}{dt^2} = s^2\frac{\partial^2 w}{\partial z^2} - \omega_{144}^2(z-u) - \omega_{244}^2(z-v) - \omega_{344}^2(z-w) \end{cases} \longrightarrow$$

$$\begin{cases} \frac{d^2\Psi}{dt^2} = c^2\frac{\partial^2\Psi}{\partial z^2} - \left(\frac{m_e c^2}{\hbar}\right)^2(\Psi-\Phi) - \left(\frac{m_g c^2}{\hbar}\right)^2(\Psi-\Lambda) - \left(\frac{m_s c^2}{\hbar}\right)^2(\Psi-\Theta) \\ \frac{d^2\Phi}{dt^2} = c^2\frac{\partial^2\Phi}{\partial z^2} - \left(\frac{m_d c^2}{\hbar}\right)^2(\Phi-\Lambda) - \left(\frac{m_b c^2}{\hbar}\right)^2(\Phi-\Theta) - \left(\frac{m_f c^2}{\hbar}\right)^2(\Phi-\Psi) \\ \frac{d^2\Lambda}{dt^2} = c^2\frac{\partial^2\Lambda}{\partial z^2} - \left(\frac{m_k c^2}{\hbar}\right)^2(\Lambda-\Theta) - \left(\frac{m_m c^2}{\hbar}\right)^2(\Lambda-\Psi) - \left(\frac{m_h c^2}{\hbar}\right)^2(\Lambda-\Phi) \\ \frac{d^2\Theta}{dt^2} = c^2\frac{\partial^2\Theta}{\partial z^2} - \left(\frac{m_n c^2}{\hbar}\right)^2(\Theta-\Psi) - \left(\frac{m_z c^2}{\hbar}\right)^2(\Theta-\Phi) - \left(\frac{m_p c^2}{\hbar}\right)^2(\Theta-\Lambda) \end{cases}$$

$$\begin{cases} \frac{d^2\Psi}{dt^2} = c^2\frac{\partial^2\Psi}{\partial z^2} - \left(\frac{m_e c^2}{\hbar}\right)^2(\Psi-\Phi) - \left(\frac{m_g c^2}{\hbar}\right)^2(\Psi-\Lambda) \\ \frac{d^2\Phi}{dt^2} = c^2\frac{\partial^2\Phi}{\partial z^2} - \left(\frac{m_d c^2}{\hbar}\right)^2(\Phi-\Lambda) - \left(\frac{m_f c^2}{\hbar}\right)^2(\Phi-\Psi) \\ \frac{d^2\Lambda}{dt^2} = c^2\frac{\partial^2\Lambda}{\partial z^2} - \left(\frac{m_m c^2}{\hbar}\right)^2(\Lambda-\Psi) - \left(\frac{m_h c^2}{\hbar}\right)^2(\Lambda-\Phi) \end{cases}$$

$$\begin{cases} \frac{\partial^2\Psi}{\partial t^2} = c^2\frac{\partial^2\Psi}{\partial z^2} - \left(\frac{m_e c^2}{\hbar}\right)^2(\Psi-\Phi) \\ \frac{\partial^2\Phi}{\partial t^2} = c^2\frac{\partial^2\Phi}{\partial z^2} - \left(\frac{m_f c^2}{\hbar}\right)^2(\Phi-\Psi) \end{cases}$$

$$\frac{\partial^2\Psi}{\partial t^2} = c^2\frac{\partial^2\Psi}{\partial z^2} - \left(\frac{m_e c^2}{\hbar}\right)^2\Psi$$

CHAIN MASS(-IN-MASS)^N

$$
\begin{cases}
\frac{d^2\Psi_1}{dt^2} = c^2\frac{\partial^2\Psi_1}{\partial z^2} - \left(\frac{m_{12}c^2}{\hbar}\right)^2(\Psi_1-\Psi_2) - \left(\frac{m_{13}c^2}{\hbar}\right)^2(\Psi_1-\Psi_3) - \left(\frac{m_{14}c^2}{\hbar}\right)^2(\Psi_1-\Psi_4) \ \ldots - \left(\frac{m_{1N-1}c^2}{\hbar}\right)^2(\Psi_1-\Psi_{N-1}) - \left(\frac{m_{1N}c^2}{\hbar}\right)^2(\Psi_1-\Psi_N) \\
\frac{d^2\Psi_2}{dt^2} = c^2\frac{\partial^2\Psi_2}{\partial z^2} - \left(\frac{m_{23}c^2}{\hbar}\right)^2(\Psi_2-\Psi_3) - \left(\frac{m_{24}c^2}{\hbar}\right)^2(\Psi_2-\Psi_4) - \left(\frac{m_{25}c^2}{\hbar}\right)^2(\Psi_2-\Psi_5) \ \ldots - \left(\frac{m_{2N-1}c^2}{\hbar}\right)^2(\Psi_2-\Psi_{N-1}) - \left(\frac{m_{2N}c^2}{\hbar}\right)^2(\Psi_2-\Psi_N) - \left(\frac{m_{21}c^2}{\hbar}\right)^2(\Psi_2-\Psi_1) \\
\frac{d^2\Psi_3}{dt^2} = c^2\frac{\partial^2\Psi_3}{\partial z^2} - \left(\frac{m_{34}c^2}{\hbar}\right)^2(\Psi_3-\Psi_4) - \left(\frac{m_{35}c^2}{\hbar}\right)^2(\Psi_3-\Psi_5) - \left(\frac{m_{36}c^2}{\hbar}\right)^2(\Psi_3-\Psi_6) \ \ldots - \left(\frac{m_{3N-1}c^2}{\hbar}\right)^2(\Psi_3-\Psi_{N-1}) - \left(\frac{m_{3N}c^2}{\hbar}\right)^2(\Psi_3-\Psi_N) - \left(\frac{m_{31}c^2}{\hbar}\right)^2(\Psi_3-\Psi_1) - \left(\frac{m_{32}c^2}{\hbar}\right)^2(\Psi_3-\Psi_2) \\
\frac{d^2\Psi_4}{dt^2} = c^2\frac{\partial^2\Psi_4}{\partial z^2} - \left(\frac{m_{45}c^2}{\hbar}\right)^2(\Psi_4-\Psi_5) - \left(\frac{m_{46}c^2}{\hbar}\right)^2(\Psi_4-\Psi_6) - \left(\frac{m_{47}c^2}{\hbar}\right)^2(\Psi_4-\Psi_7) \ \ldots - \left(\frac{m_{4N-1}c^2}{\hbar}\right)^2(\Psi_4-\Psi_{N-1}) - \left(\frac{m_{4N}c^2}{\hbar}\right)^2(\Psi_4-\Psi_N) - \left(\frac{m_{41}c^2}{\hbar}\right)^2(\Psi_4-\Psi_1) - \left(\frac{m_{42}c^2}{\hbar}\right)^2(\Psi_4-\Psi_2) - \left(\frac{m_{43}c^2}{\hbar}\right)^2(\Psi_4-\Psi_3) \\
\ldots \\
\frac{d^2\Psi_N}{dt^2} = c^2\frac{\partial^2\Psi_N}{\partial z^2} - \left(\frac{m_{N1}c^2}{\hbar}\right)^2(\Psi_N-\Psi_1) - \left(\frac{m_{N2}c^2}{\hbar}\right)^2(\Psi_N-\Psi_2) - \left(\frac{m_{N3}c^2}{\hbar}\right)^2(\Psi_N-\Psi_3) \ \ldots - \left(\frac{m_{N\,N-1}c^2}{\hbar}\right)^2(\Psi_N-\Psi_{N-1})
\end{cases}
$$

Number of masses : $N(N-1)$

	1	**2**	**3**	...	***N***
1	−	+	+	...	+
2	+	−	+	...	+
3	+	+	−	...	+
.	.	.	.	...	.
.	.	.	.	...	.
.	.	.	.	...	.
N	+	+	+	+	−

CONCLUSION

A modified mass-in-mass chain with the addition of harmonic interaction between neighboring internal masses is considered. In the case of the long-wave approximation, this chain is described by a system of equations that is a generalization of the real Klein-Gordon-Fock equation. Based on this system, we have constructed a system of equations that is a generalization of the complex – valued Klein-Gordon-Fock equation of relativistic quantum mechanics. Next, we constructed a generalization of the Dirac equation with an eight-component wave function, which has "optical" and "acoustic" dispersion branches, each with positive and negative energy. Unlike the Dirac equation with a four-component wave function, which has only an "optical" branch of dispersion, the generalized Dirac equation with an eight-component wave function has both "optical" and "acoustic" branches of dispersion, each of which is represented by branches with positive and negative energy. It is necessary to give a physical interpretation of the nature of the new solutions of the generalized Dirac equation for the optical and acoustic branches of dispersion.

We consider modified one-dimensional infinite chains of mass(-in-mass)^2. Such chain can be considered as new classes of acoustic metamaterials. The connection of the modified chains under consideration with the possibility of further generalization of the equations of relativistic quantum mechanics and with the problem of the hierarchy of fermionic masses is discussed. In the future, we plan to consider these chains for the case of equality of the characteristic frequencies of individual chains of identical masses. In the chain mass (-in-mass)^3, presumably, three optical dispersion branches can arise, which can correspond to three massive particles, within the framework of one mathematical model, which can correspond to three generations of fermions. The developed theoretical approaches may also be useful in constructing the theory of dark matter and energy.

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