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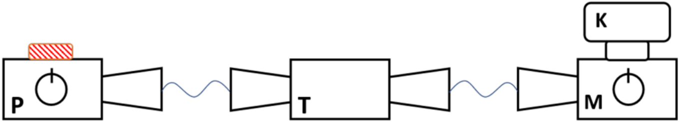
The contextual nature of quantum randomness

Quantum randomness



P-step: Preparation of quantum state (QS):

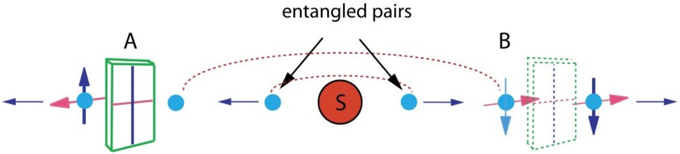
$$\hat{\rho}(0) = \hat{\rho}_0, \tag{1}$$

T-step: Evolution of QS (deterministic):

$$\dot{\hat{\rho}}(t) = \mathcal{L}\hat{\rho}(t), \tag{2}$$

M-step: Reduction of QS (random):

$$\hat{\rho}_f = \frac{\hat{M}_f \hat{\rho} \hat{M}_f^+}{tr(\hat{M}_f \hat{\rho} \hat{M}_f^+)}. \tag{3}$$



No-signaling condition (Eberhard theorem[1]):
A) $\{f_i^A\}$ - spectrum of observable F_A, detected by Alice detector D_A
B) $\{g_i^B\}$ - spectrum of observable G_B, detected by Bob detector D_B

$$\sum_j p(f_i, g_j | D_A, D_B) = p(f_i | D_A), \quad (4a)$$

$$\sum_i p(f_i, g_j | D_A, D_B) = p(g_j | D_B), \quad (4b)$$

from fundamental quantum randomness of measurement outcomes.

[1]Eberhard, P., Nuovo Cimento B, (1978), 46, 392-419.

Ontological models

Quantum contextuality conditions for ontological models[2]:
1) Quantum statistics restoration:

$$p(f) = \int d\xi d\xi' p(f|\xi,\xi')\mu_P(\xi)\gamma_T(\xi,\xi')\pi(\xi') \tag{5}$$

2) Dynamic equation for ontic variables:

$$\dot{\xi} = f_1(\xi,\xi',t), \tag{6a}$$
$$\dot{\xi}' = f_2(\xi,\xi',t), \tag{6b}$$

preserved (5) provides prediction for measurement outcome

$$\dot{\hat{\rho}}_f = \mathcal{L}_{\xi\xi'}\hat{\rho}. \tag{7}$$

N. Bohr, Phys. Rev. (1935), 48, P. 696-702.
S. Kochen, E. P. Specker, Journal of Math. and Mech. (1967), 17, p. 59-87.
R. W. Spekkens. Phys. Rev. A. (2005), 71, p. 052108.

Example: Non-selective measurement

Alice detector D_A + qubit-1 interaction

$$\hat{H}_I = \lambda \hat{F}_1 \hat{P}_A, \tag{8}$$

provides open quantum channel for qubit-1

$$\hat{\rho}_1 = \int drK(r - \lambda \hat{F}_1 t)\hat{\rho}K(r - \lambda \hat{F}_1 t) =$$

$$= \sum_i \sum_j c_i c_j |\phi_i\rangle \langle\phi_j| \int drK(r - \lambda f_i t)K(r - \lambda f_j t), \tag{9a}$$

$$\begin{pmatrix} |c_1|^2 & c_1^* c_2 & \dots & c_1^* c_N \\ c_2^* c_1 & |c_2|^2 & \dots & c_2^* c_N \\ \vdots & \vdots & \ddots & \vdots \\ c_N^* c_1 & c_N^* c_2 & \dots & |c_N|^2 \end{pmatrix} \to \begin{pmatrix} |c_1|^2 & 0 & \dots & 0 \\ 0 & |c_2|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |c_N|^2 \end{pmatrix} \tag{9b}$$

Selective stage

$$\hat{\rho}_f = \frac{\hat{M}_f \hat{\rho}_1 \hat{M}_f^+}{tr(\hat{M}_f \hat{\rho}_1 \hat{M}_f^+)},$$

$$\begin{pmatrix} |c_1|^2 & 0 & \dots & 0 \\ 0 & |c_2|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |c_N|^2 \end{pmatrix} \to \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & |c_i|^2 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \tag{10}$$

In ontological models

$$\hat{\rho}_f(\xi,\xi',t) = \int dr' n(\xi,\xi',r') K(r'-\lambda\hat{F}_1 t)\hat{\rho} K(r'-\lambda\hat{F}_1 t) =$$

$$= \frac{\hat{M}_f(\xi,\xi',t)\hat{\rho}_1 \hat{M}_f^+(\xi,\xi',t)}{tr(\hat{M}_f(\xi,\xi',t)\hat{\rho}_1(t)\hat{M}_f^+(\xi,\xi',t))}, \tag{11a}$$

$$\begin{pmatrix} |c_1|^2 & c_1^* c_2 & \dots & c_1^* c_N \\ c_2^* c_1 & |c_2|^2 & \dots & c_2^* c_N \\ \vdots & \vdots & \ddots & \vdots \\ c_N^* c_1 & c_N^* c_2 & \dots & |c_N|^2 \end{pmatrix} \to \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & |c_i|^2 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \tag{11b}$$

Ontic variables - de Broglie coordinates: $\{\xi_i\} = \mathrm{q} = (q_A, q_B)$ - measured system dBB-coordinates, $\{\xi_i'\} = \mathrm{r} = (r_1, ..., r_N)$ - detector dBB-coordinates,

$$n(r, r') = \delta(r - r'), \tag{12}$$

de Broglie coordinate dynamic - guidance equation:

$$\dot{q}_i = \frac{j_q(\mathrm{q}, \mathrm{r})}{\rho(\mathrm{q}, \mathrm{r})}, \quad \dot{r}_i = \frac{j_r(\mathrm{q}, \mathrm{r})}{\rho(\mathrm{q}, \mathrm{r})}, \tag{13}$$

then contextual protocol:

$$\hat{\rho}_f = \int dr' \delta(r(t) - r') K(r' - \lambda \hat{F} t) \hat{\rho} K(r' - \lambda \hat{F} t) =$$

$$= \frac{\hat{M}_f(r_i(t), t) \hat{\rho}_1 \hat{M}_f^+(r_i(t), t)}{tr(\hat{M}_f(r_i(t), t) \hat{\rho}_1(t) \hat{M}_f^+(r_i(t), t))}. \tag{14}$$

Momentum measurement in dBB[3][4]

Interaction Hamiltonian

$$\hat{H}_I = \lambda \hat{P}_Q \hat{P}_A, \tag{15}$$

Wave-function of MS-D_A entangled state

$$\psi(r, q, t) = \int dp' P(p') K(r - \lambda p' t) \exp\left(i \left(p' q - \frac{{p'}^2}{2m} t \right) \right), \tag{16}$$

Guidance equations for the MS and D_A

$$v_q = \frac{i\lambda}{2} \left(\frac{\psi_r^{*\prime}}{\psi^*} - \frac{{\psi_r}'}{\psi} \right) + \frac{i}{2m} \left(\frac{\psi_q^{*\prime}}{\psi^*} - \frac{{\psi_q}'}{\psi} \right), \tag{17a}$$

$$v_r = \frac{i\lambda}{2} \left(\frac{\psi_q^{*\prime}}{\psi^*} - \frac{{\psi_q}'}{\psi} \right). \tag{17b}$$

[3] A. M. Aleshin, V. V. Nikitin, P. I. Pronin, arxiv:quant-ph/2404.09934 (2024).

[4] A. M. Aleshin, V. V. Nikitin, P. I. Pronin (2023), 4, 2341511

Momentum measurement in dBB[5][6]

Superposition of n momenta

$$P(p') = A_1\delta(p' - p_1) + A_2\delta(p' - p_2) + ... + A_n\delta(p' - p_n). \quad (18)$$

Guidance equations for the MS and D_A

$$v_r = \frac{\lambda}{2}\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} A_iA_jK_iK_j(p_i + p_j)\cos\alpha_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} A_iA_jK_iK_j\cos\alpha_{ij}}, \quad (19a)$$

$$v_q = \frac{v_r}{\lambda m} + \frac{\lambda}{2}\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} A_iA_jK_iK_j\left(\frac{K_j{}'}{K_j} - \frac{K_i{}'}{K_i}\right)\sin\alpha_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{n} A_iA_jK_iK_j\cos\alpha_{ij}}, \quad (19b)$$

here

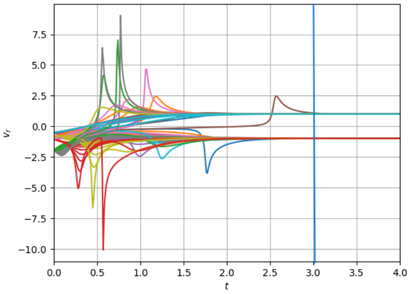
$$K_i = K\left(r - \lambda p_i t\right), \quad K_i{}' = \frac{\partial K\left(r - \lambda p_i t\right)}{\partial r},$$

$$\alpha_{ij} = \left[(p_i - p_j)q + \frac{1}{2m}(p_j^2 - p_i^2)t\right]. \quad (20)$$

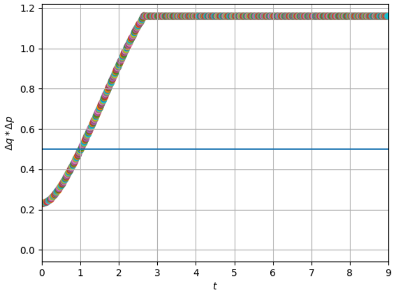
[5] A. M. Aleshin, V. V. Nikitin, P. I. Pronin, arxiv:quant-ph/2404.09934 (2024).

[6] A. M. Aleshin, V. V. Nikitin, P. I. Pronin (2023), 4, 2341511

Momentum measurement in dBB: Alice detector trajectories



(a)



(b)

Figure: a) The D_A-velocity-time dependence at momentum measurement, at different initial conditions: MS states in a superposition $|\psi_0\rangle = A_1 |\rightarrow\rangle + A_2 |\leftarrow\rangle$ at probabilities $A_1^2/A_2^2 = 1/9$. Parameters: $\lambda = 1$, $m = 1$. b) The dependence of the product $\Delta p \Delta q$ on time during momentum measurement. The thin line marks the lower limit $\hbar/2 = 0.5$. Parameters: $m = 1000$, $\lambda = 1$, $p_1 = -5, ..., p_{11} = 5$.

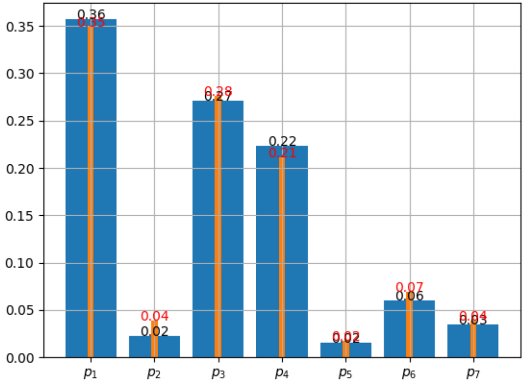


Figure: The momentum probability distribution obtained by the numerical simulation of the measurement at $A_1/A_2/A_3/A_4/A_5/A_6/A_7 = 9/3/8/7/2/4/3$. Blue columns – the numerical result, Orange lines – theoretical predictions.

Momentum measurement in dBB: Contextuality

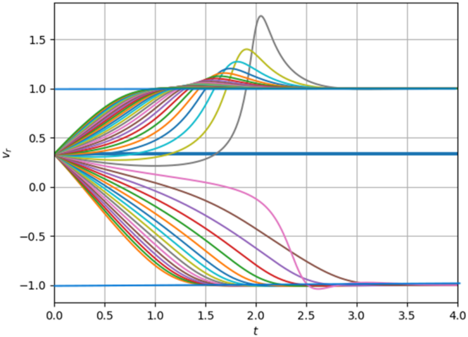
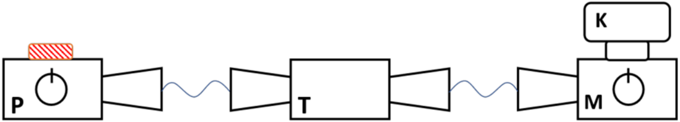


Figure: The D_A-velocity-time dependence at momentum measurement, at different D_A-initial conditions $\{r_0\}$ for case $p_1 = -1, p_2 = 1$, $A_1^2/A_2^2 = 1/4$, $\lambda = 1$, $m = 1$ with the fixing initial condition $q = 0$ for the measured system. Measurement outcomes depend on initial states of D_A-dB-coordinates only. It is the quantum contextuality. In the classical limit (bold line) the contextuality is absent.

Conclusions: Experimental restrictions



If you know D_A+MS-ontic variables $\{\xi_i\}$:
1) initial values $\{\xi_0\}$,
2) dynamics $\xi_i(t)$,
then you can predict observable value f (11) and ruin no-signaling condition (4).
But in normal conditions:
1) you need information about $(N \sim N_A)$ D_A+MS-ontic variables $\{\xi_i\}$[7]
2) you have uncontrollable stochastic noise.

[7] G. Tastevin, F. Laloe, Comptes Rendus. Physique, 22(1):99-116, (2021).

Wigner-function[8]

$$f_N(\vec{p},\vec{q},t) = \frac{1}{(2\pi)^3}\int \rho\left(\vec{q}-\frac{\vec{y}}{2},\vec{q}+\frac{\vec{y}}{2},t\right)e^{i\vec{p}\vec{y}}d\vec{y}, \qquad (21)$$

dynamic equation for f_N - Moel equation[9]:

$$\frac{\partial f_N}{\partial t} + \{f_N, H\}_M = 0, \qquad (22)$$

here $\{f_N, H\}_M = f_N \star H - H \star f_N$ - Moel bracket, where

$$f_N \star H = \frac{2}{\hbar}\exp\frac{\hbar}{2}\left[\frac{\partial}{\partial\vec{p}_f}\frac{\partial}{\partial\vec{q}_H} - \frac{\partial}{\partial\vec{p}_H}\frac{\partial}{\partial\vec{q}_f}\right]H(\vec{p},\vec{q})f_N(\vec{p},\vec{q},t), \qquad (23)$$

[8] E. Wigner. Phys. Rev., 40:749-759, (1932).
[9] J. E. Moyal. Math. Proc. of the Cambr. Phil. Soc., 45(1):99-124, (1949).

Hydrodynamic system

Dynamic equation for f_1 - Moel-Boltsman equation:

$$\frac{\partial f_1}{\partial t} + \{f_1, H\}_M = F_{col}, \tag{24}$$

after projection on x-space we have hydrodynamic system[10]

$$\begin{aligned}
\frac{\partial n}{\partial t} + \frac{\partial (np_1^i)}{\partial q^i} &= 0, \\
\frac{\partial (np_1^i)}{\partial t} + \frac{\partial (np_1^i p_1^j + P^{ij})}{\partial q^j} + n\frac{\partial U}{\partial q_i} &= D_1, \\
\frac{\partial (n\theta)}{\partial t} + \frac{\partial (np_1^i \theta)}{\partial q^i} + \frac{2}{3}\left(\frac{\partial u^i}{\partial q^i} - D_{ij}P^{ij}\right) &= D_2,
\end{aligned} \tag{25}$$

with state equation

$$P_q = P_q(n, \theta_q), \tag{26}$$

[10] T. Takabayasi. Prog. of Th. Physics, 11(4-5):341-373, 04, (1954).

Pure state: de Broglie-Bohm theory

In the pure state case[11]

$$\hat{\rho}^2 = \hat{\rho}, \tag{27}$$

hydrodynamic system - de Broglie guidance equations:

$$p_1^i = \partial^i S,$$
$$\frac{dp_1^i}{dt} = \frac{1}{2m}\frac{\partial}{\partial q_i}\left(\frac{\partial^2\sqrt{n}}{\sqrt{n}}\right) - \frac{\partial U}{\partial x_i}, \tag{28}$$

with ideal gas - state equation

$$P_q^{ij} = \frac{1}{4}\partial^i\partial^j \ln n$$
$$P_q = n\theta_q. \tag{29}$$

[11] T. Takabayasi. Prog. of Th. Physics, 11(4-5):341-373, 04, (1954).

Wigner-Boltzmann formalism: momentum measurement

D_A - Classical detector with ontic coordinates $\{r_i\}$,
MS - quantum measured system with ontic coordinates $\{q_i\}$.
Moel equation for f_2

$$\frac{\partial f_2}{\partial t} + \left(\frac{p}{m} + \lambda k\right)\frac{\partial f_2}{\partial q} + \left(\frac{k}{M} + \lambda p\right)\frac{\partial f_2}{\partial r} = 0, \qquad (30)$$

Hydrodynamic system

$$\begin{gathered}
\frac{\partial n}{\partial t} + \frac{1}{m}\frac{\partial((p_1 + m\lambda k)\, n)}{\partial q} + \frac{1}{M}\frac{\partial((k_1 + M\lambda p_1)n)}{\partial r} = 0, \\
\frac{\partial(np_1)}{\partial t} + \frac{\partial(np_1^2 + P)}{\partial q} + \frac{\partial(np_1 k_1)}{\partial r} = \frac{\partial(\lambda p_1 k_1 n)}{\partial q}, \\
\frac{\partial(nk_1)}{\partial t} + \frac{\partial(nk_1^2)}{\partial r} + \frac{\partial(np_1 k_1)}{\partial q} = \frac{\partial(\lambda k_1 p_1 n)}{\partial r}, \\
\frac{\partial(n\theta)}{\partial t} + \frac{\partial(np_1\theta + u)}{\partial q} + \frac{\partial(nk_1\theta)}{\partial r} - DP = \frac{\partial(\lambda k_1 \theta n)}{\partial r}.
\end{gathered} \qquad (31)$$

THANK YOU FOR YOUR ATTENTION!