Finite system size corrections to the effective coupling in ϕ^4 scattering

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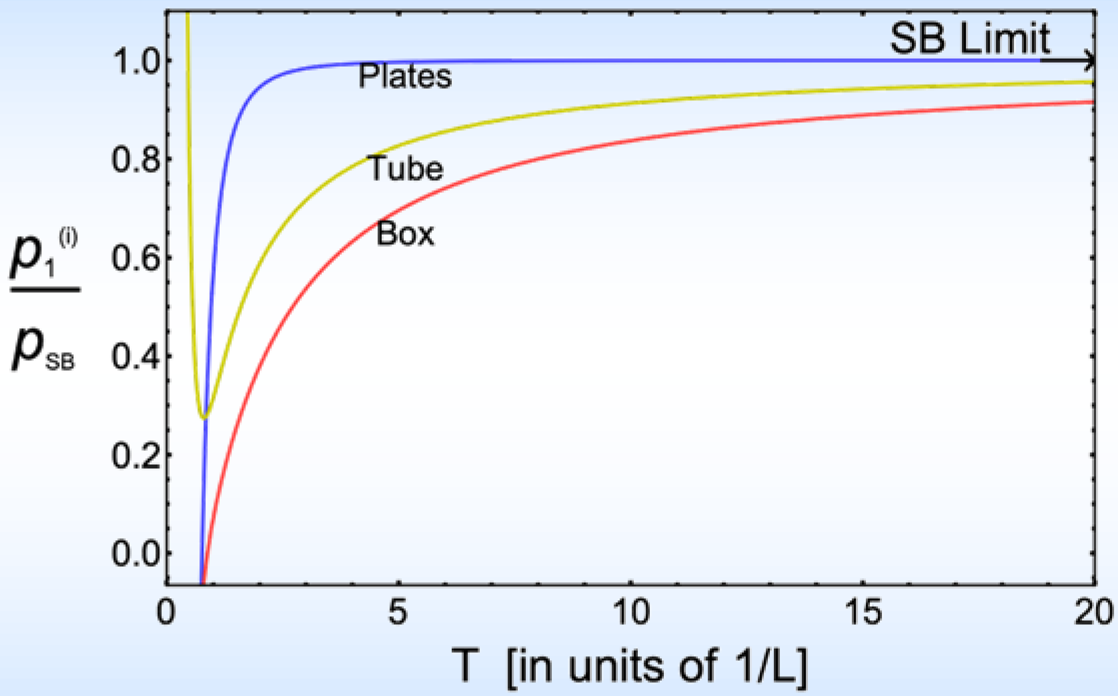
WAH and Du Plessis, Phys.Rev.D 105 (2022) 9, L091901 [2203.01259]
WAH and Du Plessis, Phys.Rev.D 109 (2024) 3, 036013 [2308.08651]

Motivation: Thermodynamics of Small Systems

Does Finite Size Affect Thermodyn.?

- Test using free scalar field theory



~10% effect at 400 MeV for ~1 fm system

Mogliacci, Kolbé, and WAH, PRD102 (2020)

  – p decreases as T decreases for fixed L, converging to usual T = 0 Casimir effect

Finite Size Effects on Trace Anomaly

- Trace anomaly Δ
  - gives speed of sound
  - important for η/s extraction in QCD
  - due to running coupling
    - finite size breaks conformal invariance, but $\Delta = 0$ at LO
- Non-trivial conceptual issues for QCD:
  - How to regularize and renormalize?
    - Dim reg difficult to generalize in finite size setting
  - Torons
- First examine finite size effects for running coupling in 2 => 2 scattering in ϕ^4 theory

Denominator Regularization

Running Coupling

- Requires NLO loop calculation
  - Thus requires a regularization scheme
    - Which scheme to use?
- All current schemes have drawbacks:
  - Pauli-Villars breaks gauge invariance
  - Zeta breaks BRS invariance
  - Dim reg
    - Inappropriate for finite size system:
      - I know how to perform *n* sums, where *n* is a positive integer; how do I perform $3 - \epsilon$ sums?
      - Dim reg requires symmetry in all directions; what if the system is asymmetric?

Define New Regularization Scheme

- Denominator regularization (den reg) instead of dimensional regularization (dim reg)
  - Keep number of dimensions fixed
  - Feynman *x* combine propagators
  - Analytically continue the power of the single denominator
  - Introduce fictitious scale μ to maintain dimensions
  - Allow for coefficient functions $f_{(n,p)}(\epsilon)$, $f_{(n,p)}(0) = 1$, that depend on original denominator power *n* and the superficial degree of divergence *p*
  - E.g.:

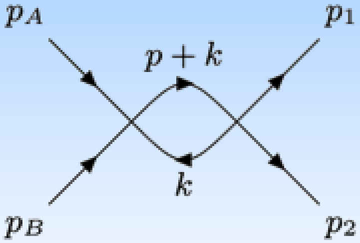
$$V(p^2;\mu,\epsilon) = -\frac{1}{2}\int_0^1 dx \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{i\mu^\epsilon}{(k^2-\Delta^2)^2}$$

$$\Rightarrow -\frac{1}{2}\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{i f_{(2,0)}(\epsilon)(-\mu^2)^{2\epsilon}}{(k^2-\Delta^2)^{2+\epsilon}}$$

Advantages of Den Reg

- As easy to implement as dim reg
- Maintains Lorentz invariance
- Internally consistent:
  - Momenta and fields all in reps of SO(1,n)
    - Cf dim reg, where momenta are in SO(1,n-ϵ) but fields are in SO(1,n)
- Fixed dim's:
  - Well suited for thermal field theory, field theory in a finite-sized system, field theory in curved space
  - γ^5 and $\epsilon^{\mu\nu\rho\sigma}$ uniquely well defined
    - Manifestly correct prediction of axial anomaly? (Dim reg must break Lorentz inv. to get axial anom.)
  - should maintain SUSY (to check; cf dim reg)

Examples: ϕ^4 Theory to NLO



$$= \quad V \to i\frac{(-\mu^2)^\epsilon f_{(2,0)}(\epsilon)}{2}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{1}{(k^2-\Delta^2)^{2+\epsilon}}$$

$$\Delta^2 \equiv m^2 - x(1-x)p^2 - i\varepsilon, \qquad p \equiv p_1 + p_2$$

– Need the integral $\displaystyle\int k_E^3 dk_E \frac{\mu^{2\epsilon}}{(k_E^2+\Delta^2)^{2+\epsilon}} = \frac{1}{2}\frac{1}{\epsilon(1+\epsilon)}\left(\frac{\mu^2}{\Delta^2}\right)^\epsilon$

– Only $1/\epsilon$ divergence already, set $f_{(2,0)}(\epsilon)\equiv 1$

– Then

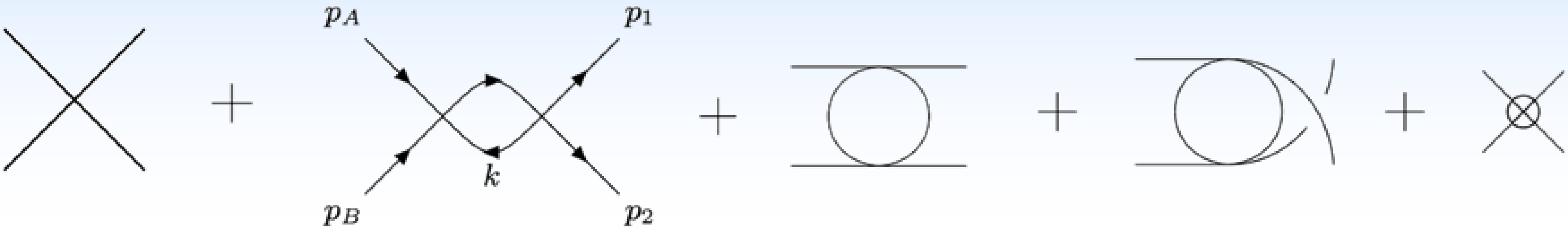
$$V(p^2;\,\mu) = -\frac{1}{2(4\pi)^2}\int_0^1 dx\left[\frac{1}{\epsilon} - 1 + \ln\left(\frac{\mu^2}{\Delta^2}\right)\right]$$

- Cf dim reg, -1 => $-\gamma_E + \ln(4\pi)$
- Den reg satisfies unitarity in inf. vol. limit

Finite Size Effects on Running Coupling in Continuum Field Theory

2 => 2 at NLO in ϕ^4 in Finite System

- Feynman Diagrams:



- Define $(-i\lambda)^2 iV(p^2) \equiv$ 

- Impose Periodic B.C.'s

$$V(p^2, \{L_i\}; \mu, \epsilon) = -\frac{1}{2}\int_0^1 dx \int \frac{dk^0}{2\pi} \sum_{\vec{k}\in\mathbb{Z}^3} \frac{1}{(2\pi)^3 L_1 L_2 L_3} \frac{\mu^{2\epsilon}}{[k^2 - \Delta^2]^{2+\epsilon}}$$

Capture the Divergence

- Result is a generalized Epstein Zeta fcn

$$V(p^2,\{L_i\};\mu,\epsilon) = -\frac{1}{2}\frac{1}{2\pi}\frac{1}{(2\pi)^3 L_1 L_2 L_3}\frac{\sqrt{\pi}\Gamma\left(\frac{3}{2}+\epsilon\right)}{\Gamma(2+\epsilon)}\int_0^1 dx \sum_{\vec{k}\in\mathbb{Z}^3}\frac{\mu^{2\epsilon}}{\left(\sum_{i=1}^3\left(\frac{k^i}{L_i}+xp^i\right)^2+\Delta^2\right)^{\frac{3}{2}+\epsilon}}$$

- Poisson Summation Formula: $\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p}\tilde{F}(\vec{m})$

- Yields new analytic continuation for g.E.Z:

$$\sum_{\vec{n}\in\mathbb{Z}^p}(a_i^2 n_i^2 + b_i n_i + c - i\varepsilon)^{-s} = \frac{1}{a_1\cdots a_p}\frac{1}{\Gamma(s)}\Bigg[\pi^{p/2}\Gamma(s-\frac{p}{2})\Big(c-\sum\frac{b_i^2}{4a_i^2}-i\varepsilon\Big)^{\frac{p}{2}-s}$$

$$+2\pi^s\sum_{\vec{m}\in\mathbb{Z}^p}{}' e^{-2\pi i\sum\frac{m_i b_i}{2a_i^2}}\left(\frac{c-\sum\frac{b_i^2}{4a_i^2}-i\varepsilon}{\sum\frac{m_i^2}{a_i^2}}\right)^{\frac{p}{4}-\frac{s}{2}} K_{s-\frac{p}{2}}\left(2\pi\sqrt{(c-\sum\frac{b_i^2}{4a_i^2}-i\varepsilon)(\sum\frac{m_i^2}{a_i^2})}\right)\Bigg]$$

Finite Size Result at NLO

- Found the pole! And the F.S. correction!

$$V(p^2,\{L_i\};\mu,\epsilon) = -\frac{1}{2}\frac{1}{(4\pi)^2}\int_0^1 dx \left\{\frac{1}{\epsilon} - 1 + \ln\frac{\mu^2}{\Delta^2} + 2\sum_{\vec{m}\in\mathbb{Z}^3}{}' e^{-2\pi i x \sum m_i p^i L_i} K_0\left(2\pi|\Delta|\sqrt{\sum m_i^2 L_i^2}\right)\right\}$$

  – Correction
    - goes to 0 as L_i, p => infinity
    - satisfies unitarity/optical theorem
- Optical thm check highly non-trivial
  – Requires generalization of a number theory result from Hardy/Ramanujan

Numerical Results for V in 1D Compact

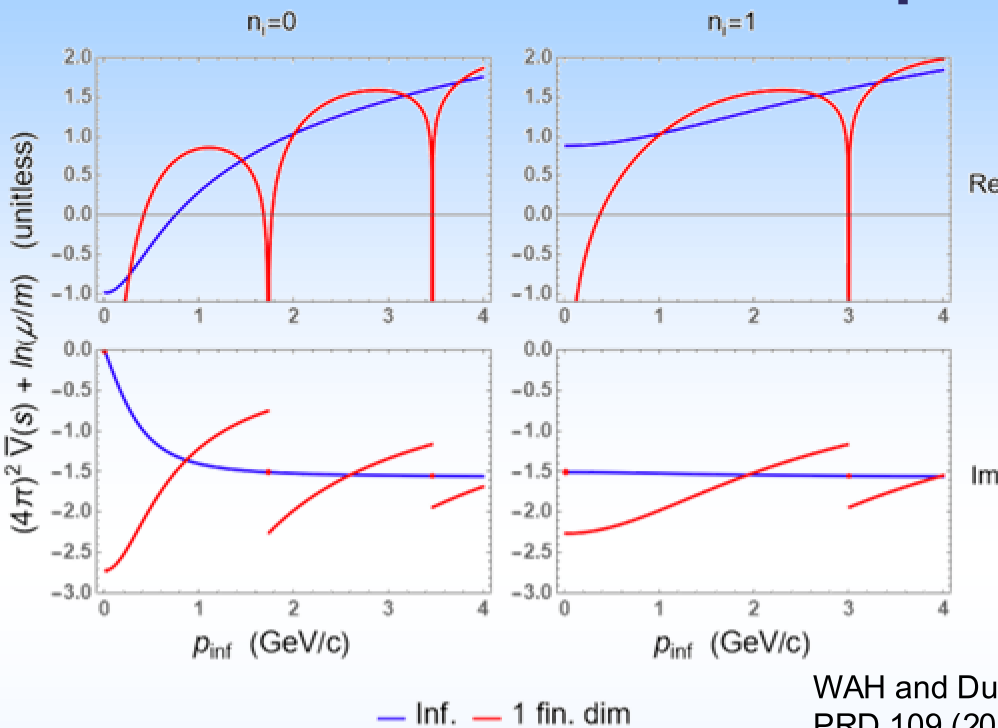
- s Channel (p):

$$p^i_{\text{in}} = \left(\frac{n_i}{L}, 0, p_{\text{inf}}\right)^i$$

$$p^i_{\text{out}} = \left(\frac{n_f}{L}, \sin(\theta) p_{\text{inf},\,f}, \cos(\theta) p_{\text{inf},\,f}\right)^i$$

$$p_{\text{inf},\,f} \equiv \sqrt{p^2_{\text{inf}} + \frac{n_i^2 - n_f^2}{L}},$$

n_i is mode in finite direction



WAH and Du Plessis, PRD 109 (2024)

  - V(p=>∞) converges to infinite volume result almost everywhere
  - V(p>0) => -∞ are "geometric bound states"
    - All outgoing momentum in the finite direction
    - Outgoing particle "gets stuck" in the geometry
  - V(p=>0) IR divergence is desired finite size effect

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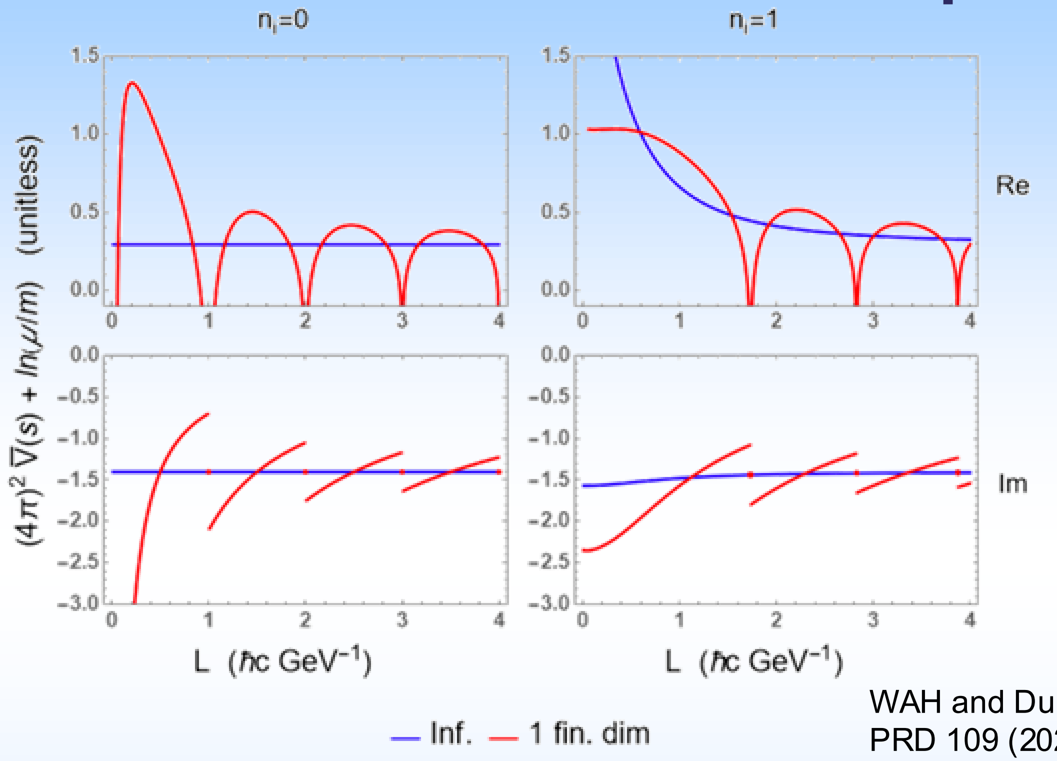
- s Channel (L):



WAH and Du Plessis, PRD 109 (2024)

  - Similar to p dependence:
    - V(L=>∞) converges to infinite volume result almost everywhere
    - V(L>0) => -∞ are "geometric bound states"
      - All outgoing momentum in the finite direction
      - Outgoing particle "gets stuck" in the geometry
    - V(L=>0) IR divergence is desired finite size effect
  - Nontrivial L => 0 dependence for n_i = 1 induced by n_i/L momentum in the finite direction

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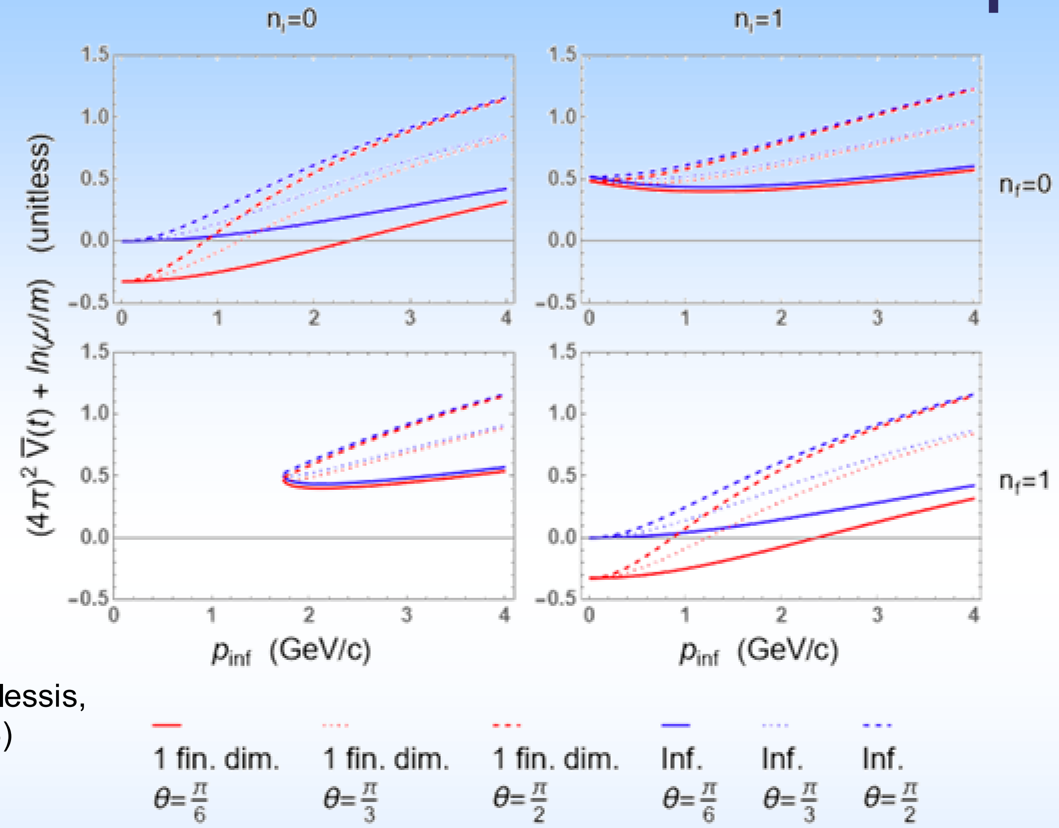

WAH and Du Plessis, PRD 109 (2024)

  - Curves end for $n_i = 0$ and $n_f = 1$ as not enough momentum in system
  - Non-zero mass acts as IR regulator, so V(p=>0) is finite
  - V(p=>∞) converges to infinite volume result

Numerical Results for V in 1D Compact

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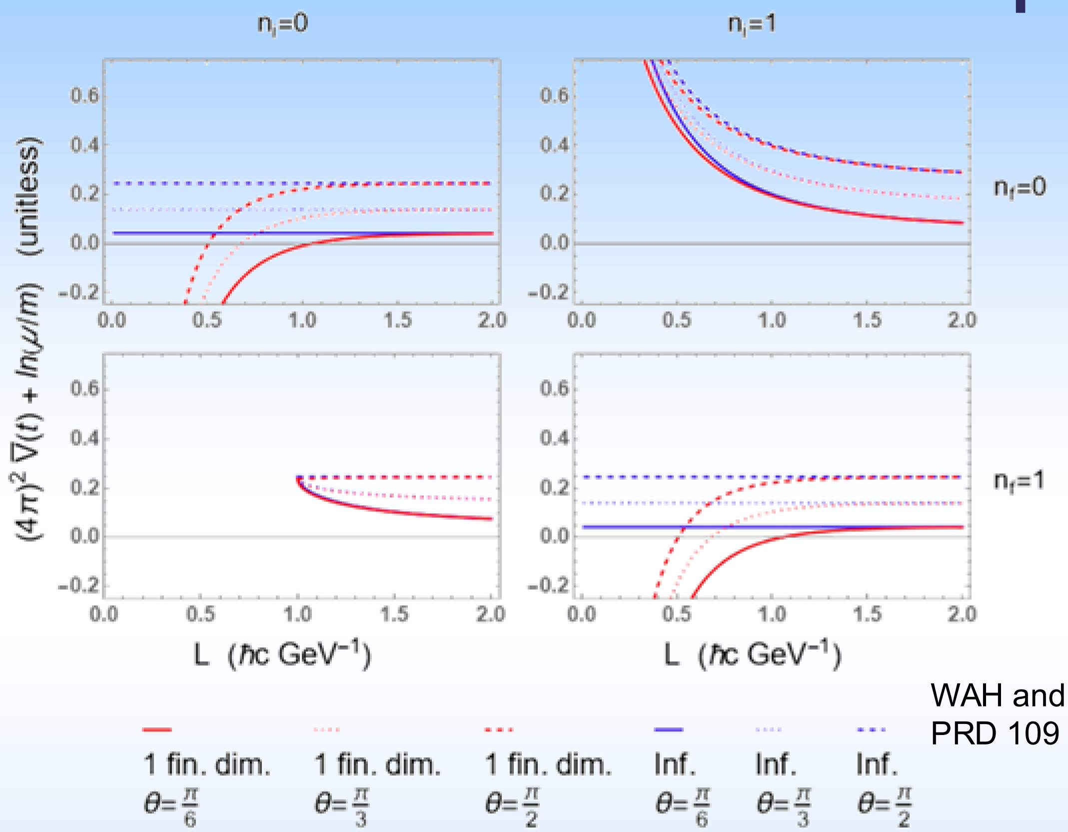


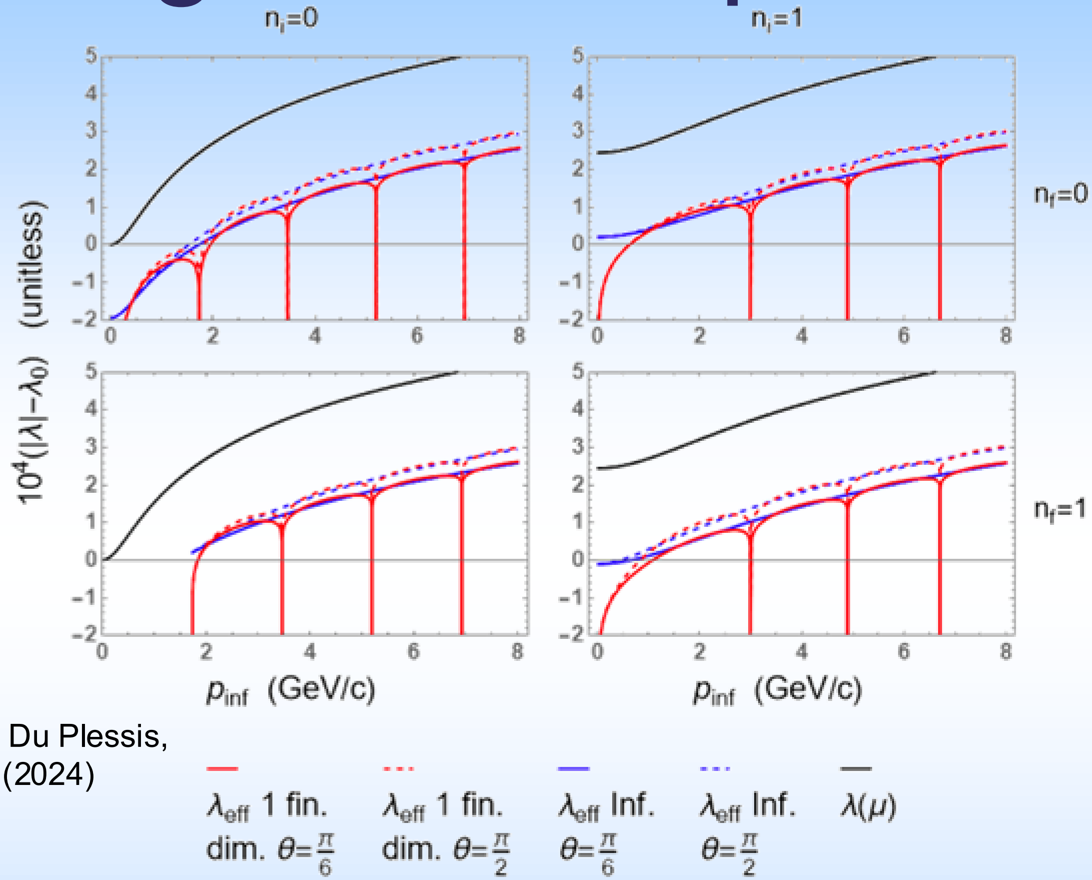
WAH and Du Plessis, PRD 109 (2024)

  - Curves end for $n_i = 0$ and $n_f = 1$ as not enough momentum in system
  - V(L=>0) diverges for systems with available momenta
  - V(p=>∞) converges to infinite volume result

Effecive Coupling in 1 Compact D

- $\lambda_0 = 0.1$; $\lambda(p)$

Black: Callan-Symanzik running coupling
Blue: infinite volume effective coupling
Red: finite-size effective coupling



WAH and Du Plessis, PRD 109 (2024)

- Running coupling captures leading log of effective coupling
- p => ∞ finite size effective coupling converges almost everywhere to infinite volume
- Notice influence of geometric bound states
  - λ => 0 at bound state poles
- p => 0 demonstrates finite size effects, where λ => 0

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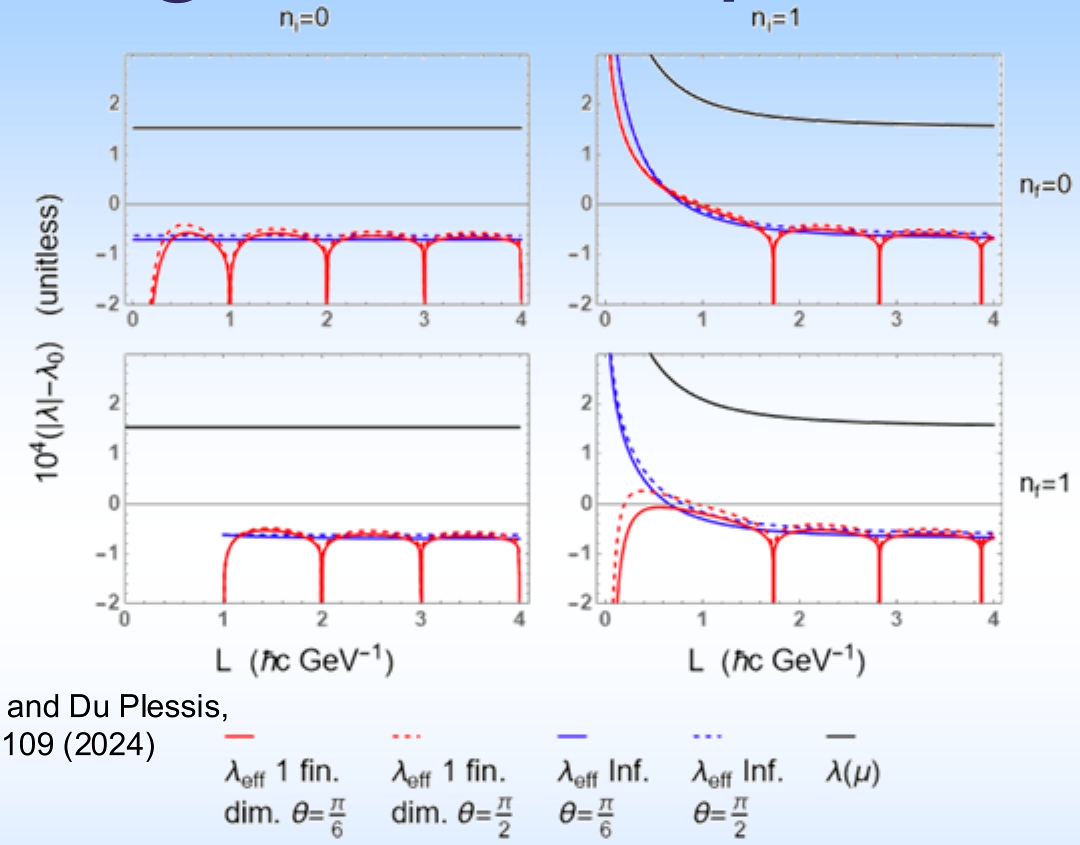

WAH and Du Plessis,
PRD 109 (2024)

- Running coupling captures leading log of effective coupling
- L => ∞ finite size effective coupling converges almost everywhere to infinite volume
- Notice influence of geometric bound states
  - λ => 0 at bound state poles
- L => 0 demonstrates finite size effects, where λ => 0
  - Surprising, as one expects $p_{typ} \sim 1/L \Rightarrow \infty$ and $\lambda \Rightarrow \infty$ from positive beta function for ϕ^4

Conclusions

- Small systems are a fascinating frontier in physics
- Placed massive, free scalar field in a box
  - Thermodynamics significantly altered; mimics QCD
- Introduced novel regularization scheme, den reg
  - Applicable in finite systems, curved spacetimes, thermal field theory; preserves Lorentz invariance; passes consistency checks; preserves QED Ward, SUSY? predicts axial anomaly?
- Computed NLO scattering in ϕ^4 in 1, 2, and 3 compact dim's
  - Analytic continuation of the generalized Epstein zeta function
  - Checked unitarity
  - Discovered geometric bound states
  - Captured finite size corrections
  - Computed finite size effects in effective coupling:
    - $\lambda(p \Rightarrow 0) \Rightarrow 0$, consistent with ϕ^4 beta function
    - $\lambda(L \Rightarrow 0) \Rightarrow 0$, seemingly inconsistent with ϕ^4 beta function
- A lot of interesting work to do!!

Bonus Slides

All from WAH and du Plessis,
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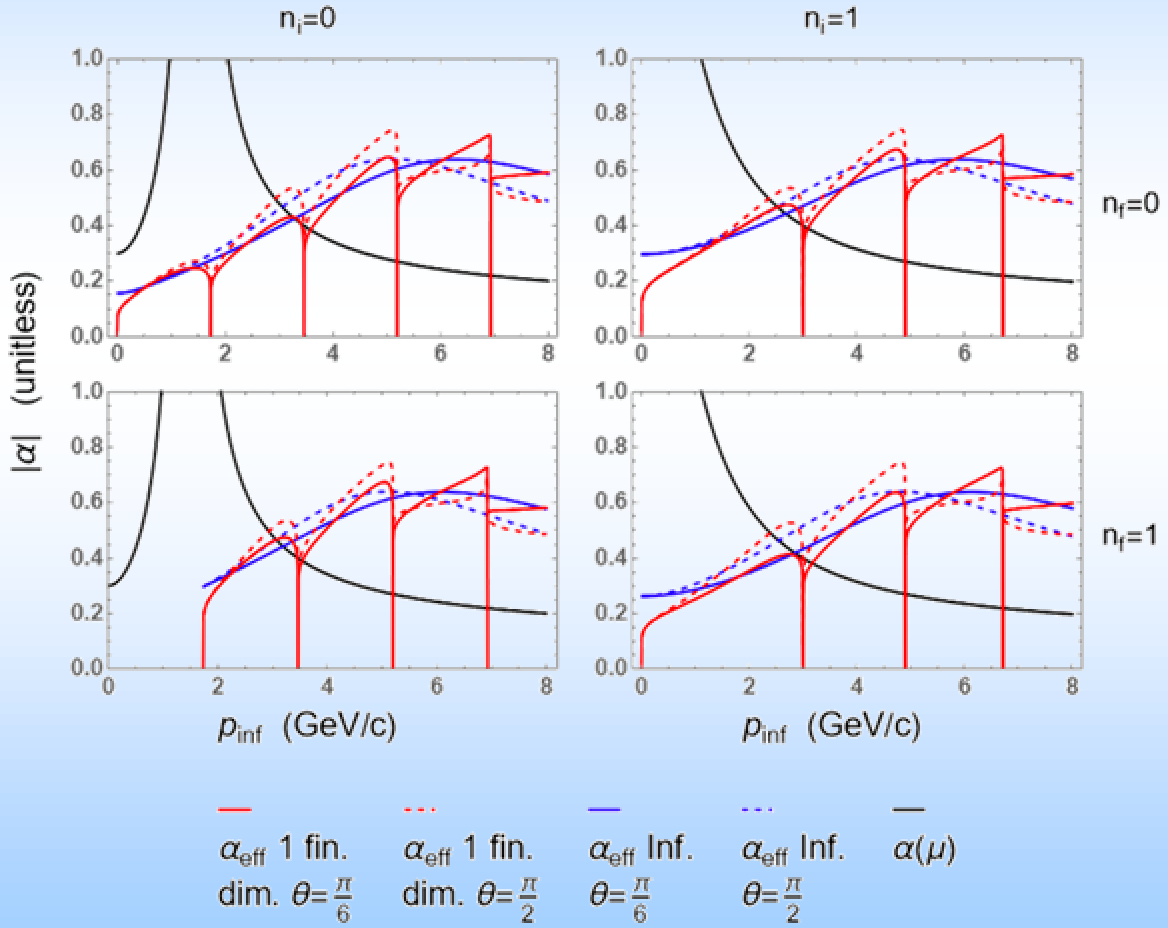
Running Coupling in 1 Compact D

- α_0 = 0.3; α(p)



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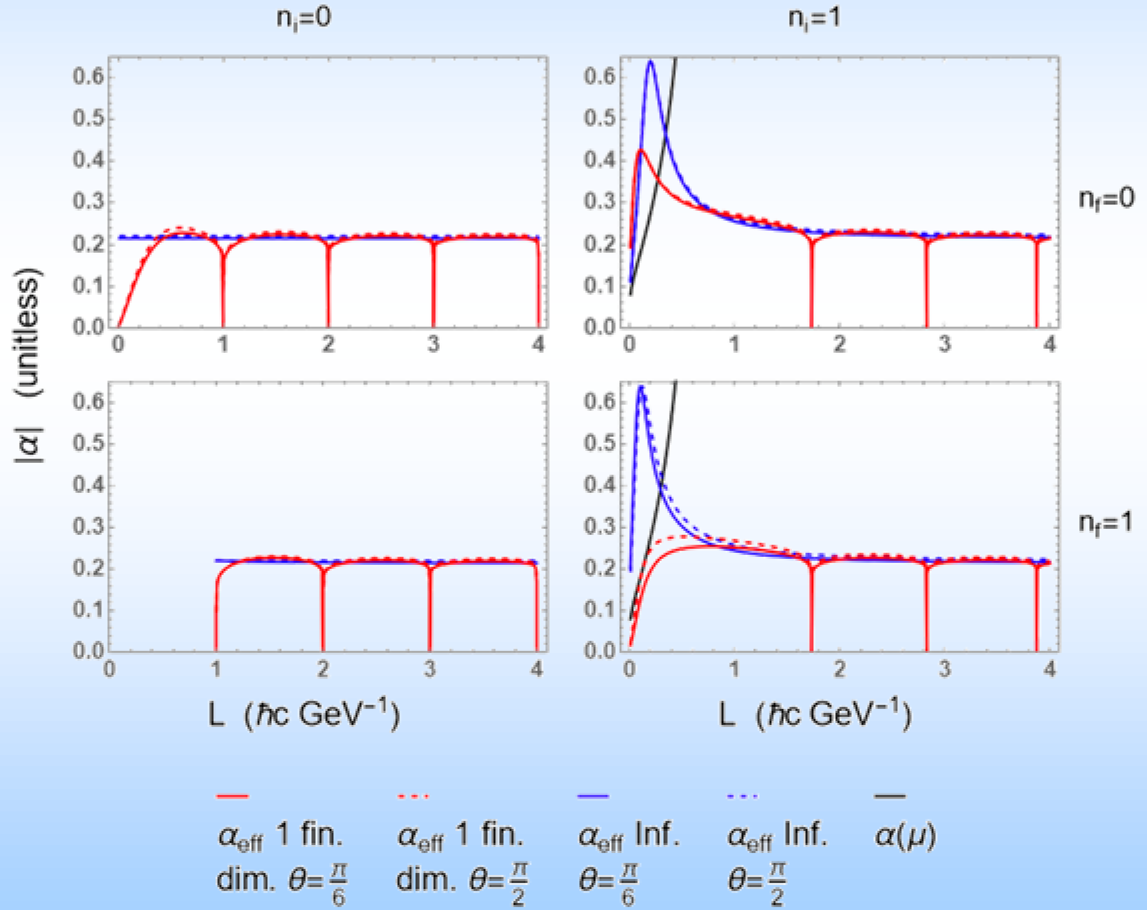

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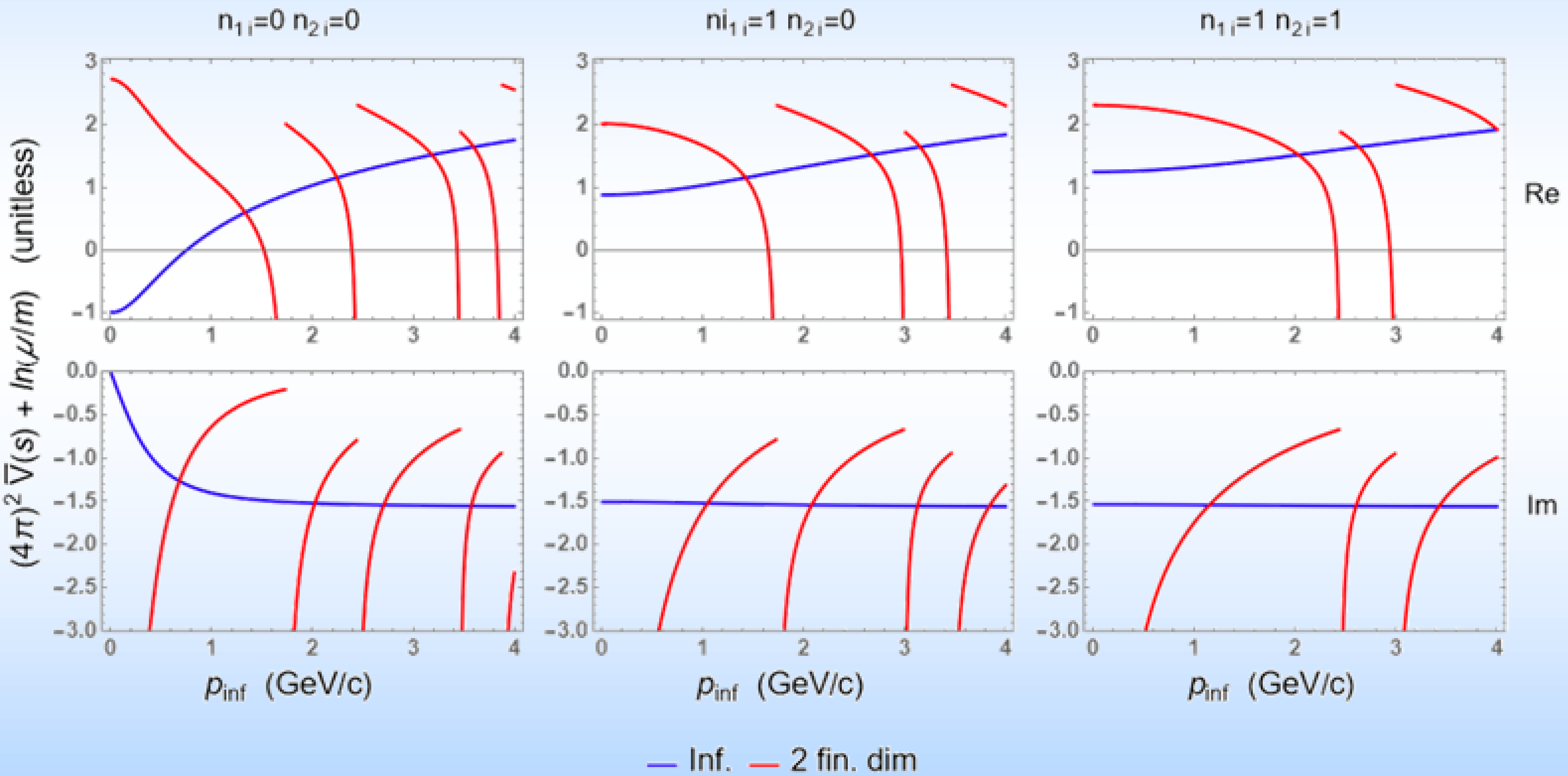

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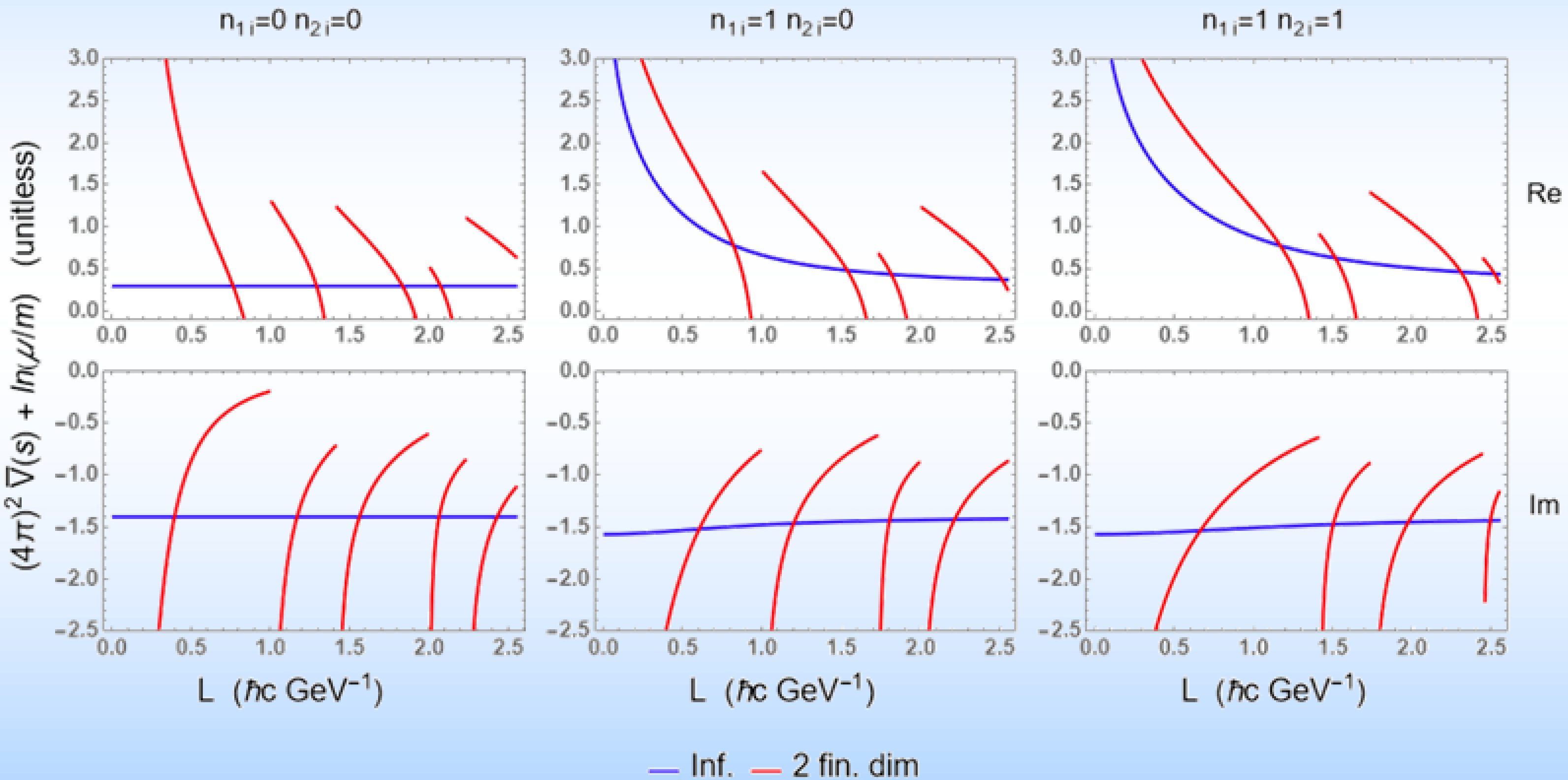

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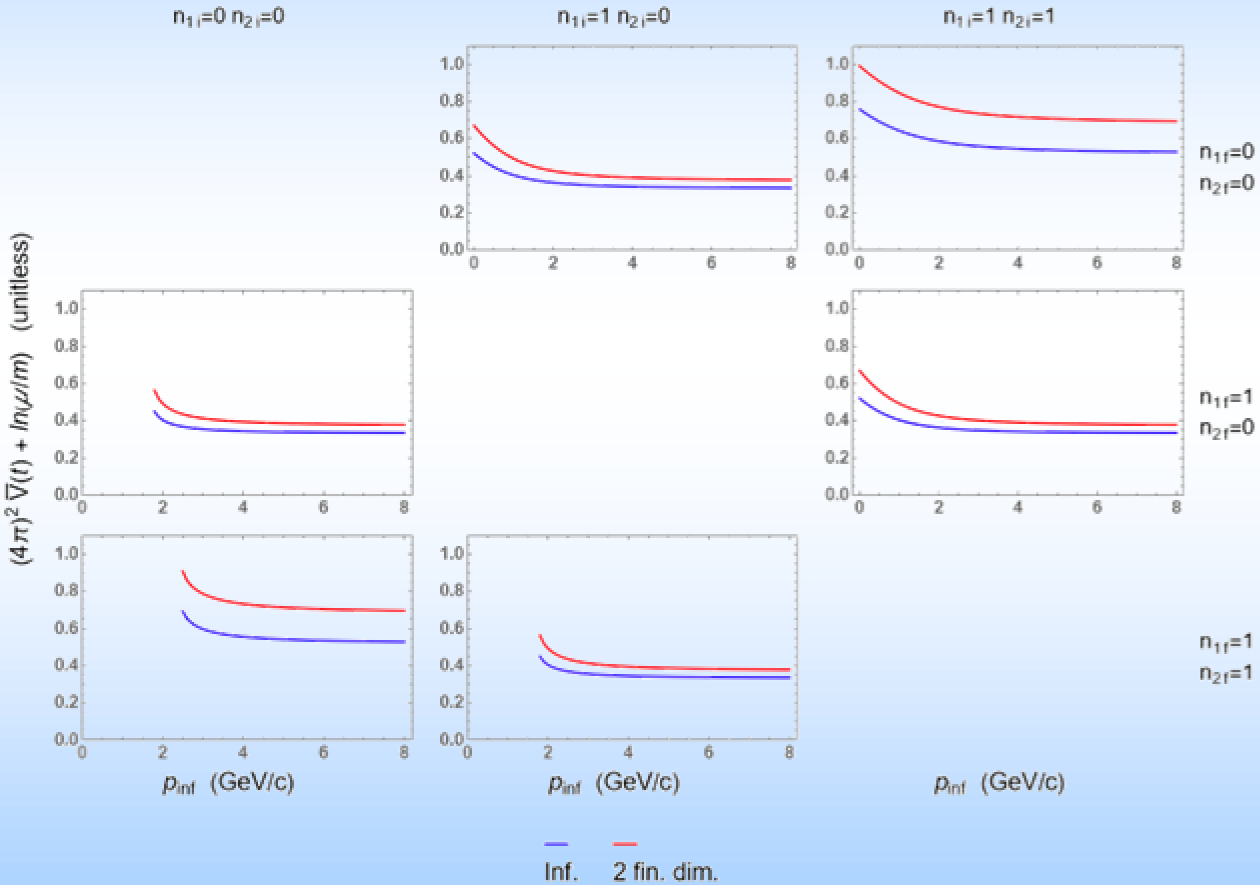

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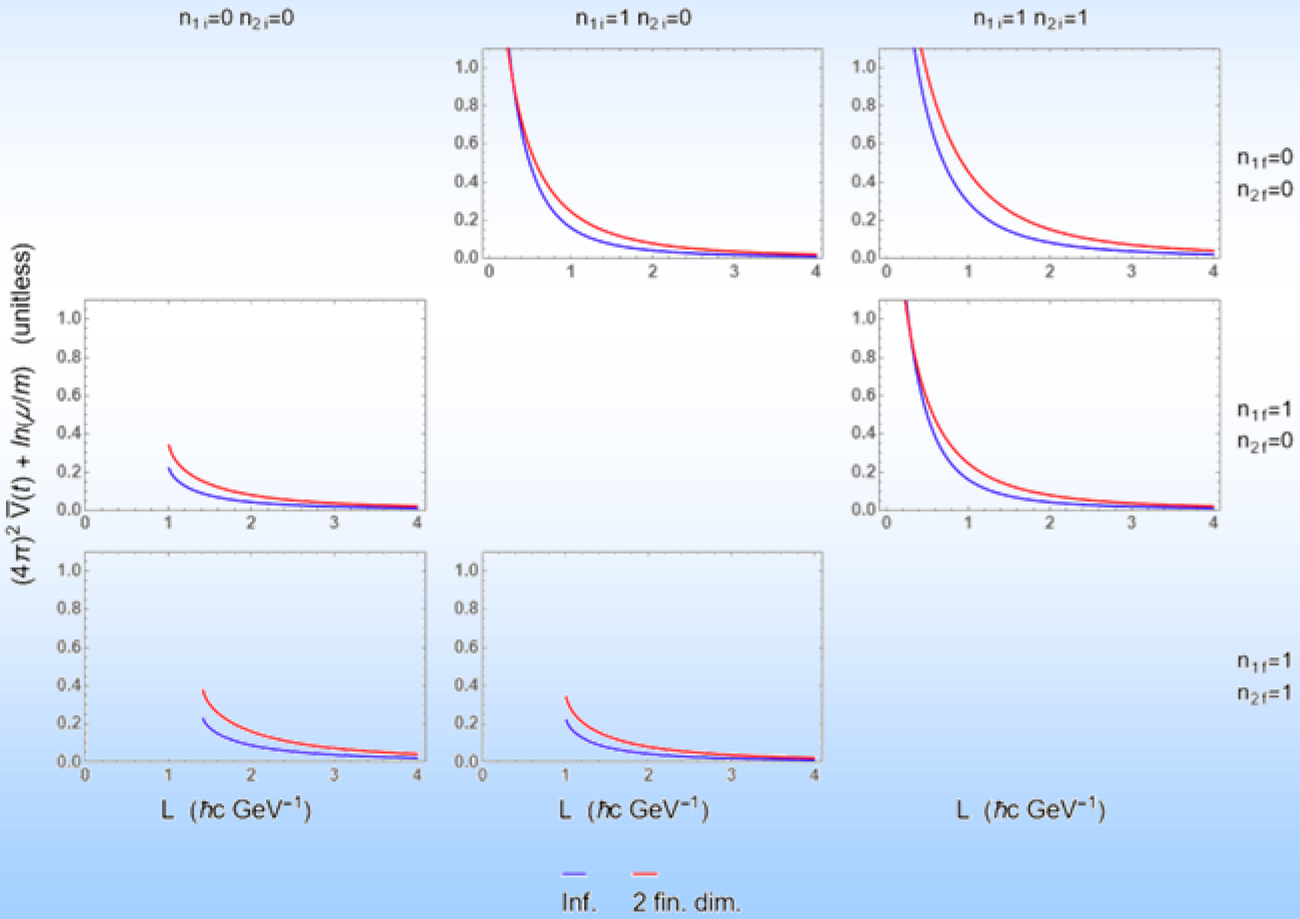
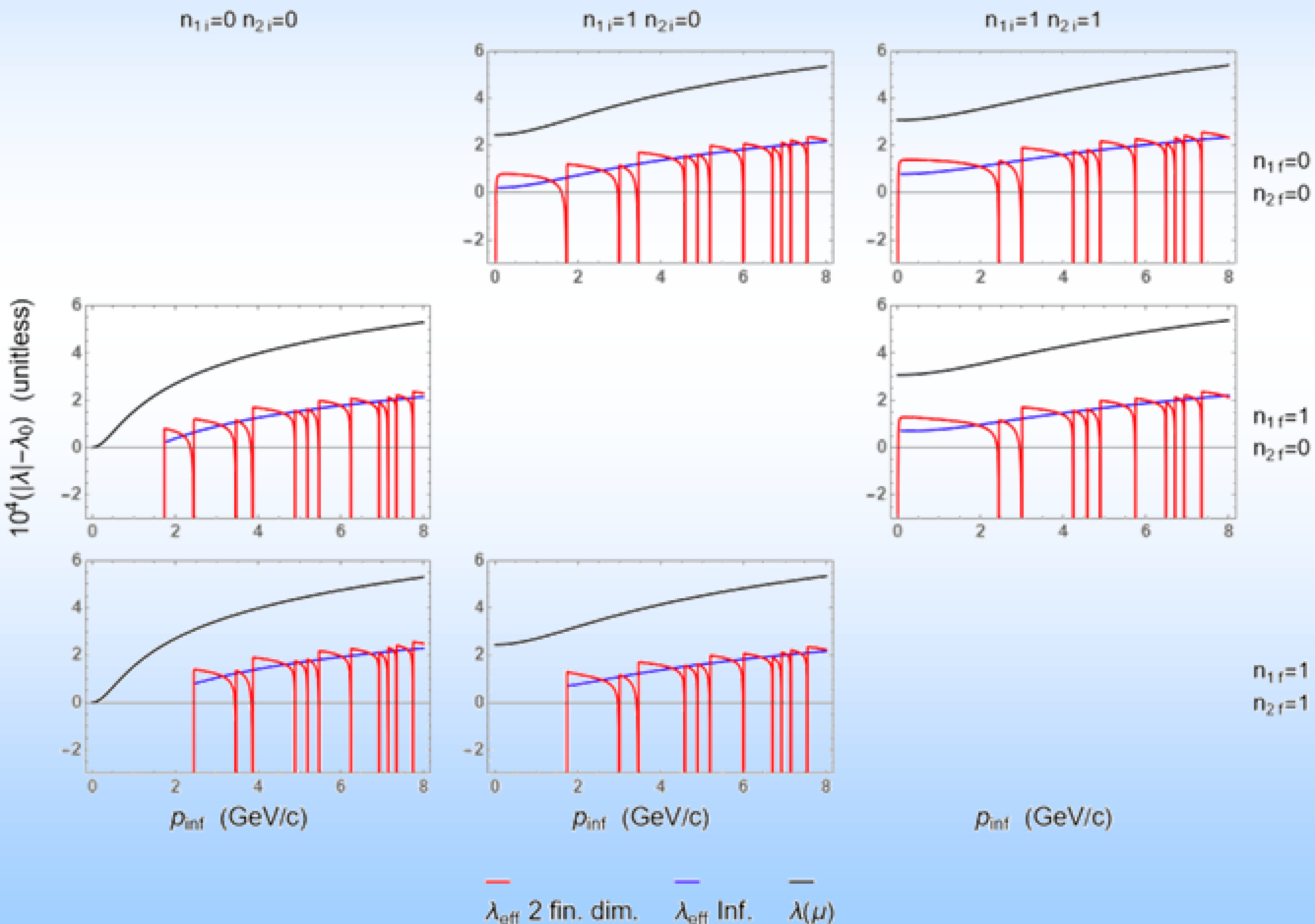


Effecive Coupling in 2 Compact D

- $\lambda_0 = 0.1; \lambda(p)$



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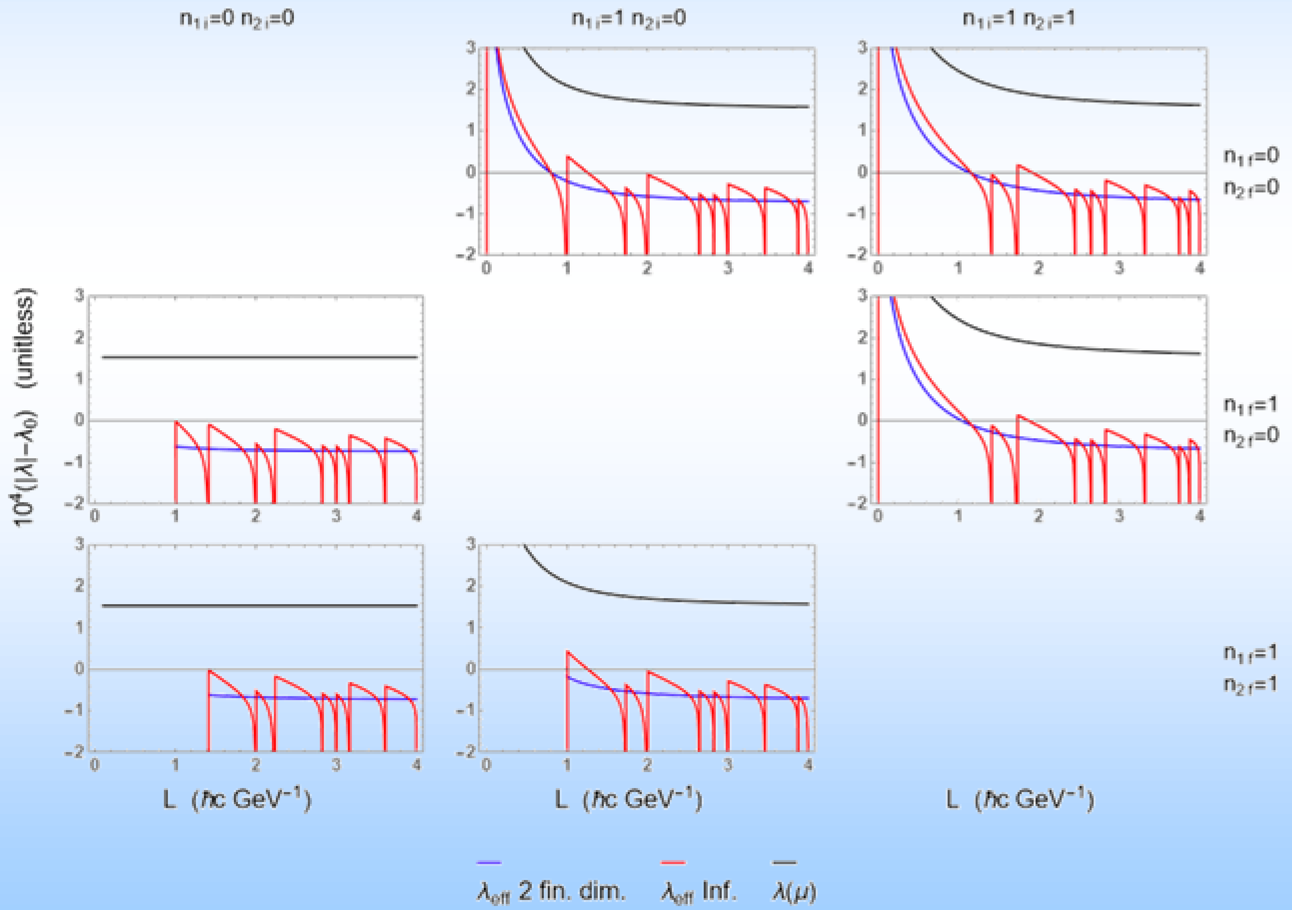

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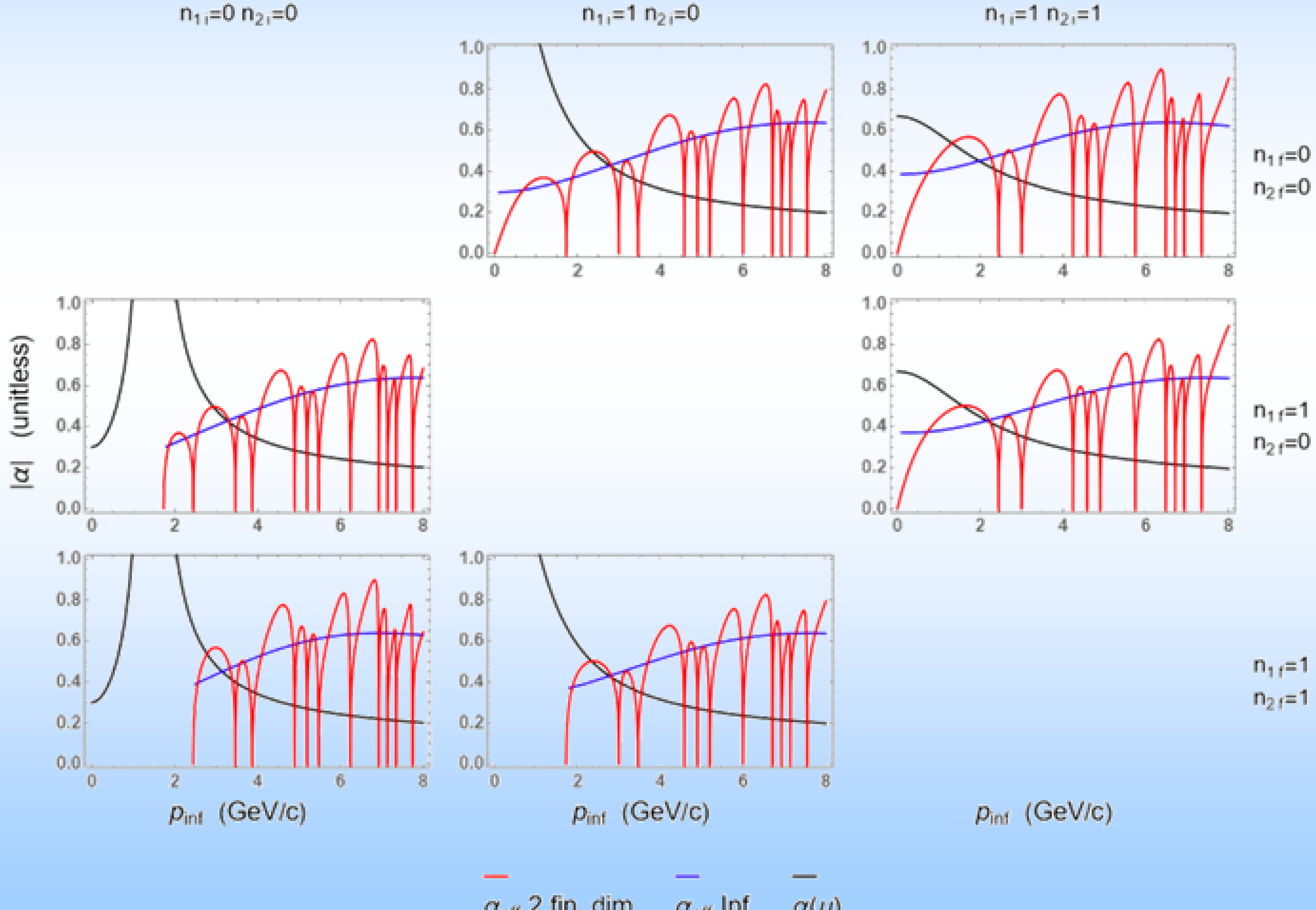

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