Chern-Simons boundary layers in the Casimir effect

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In this talk

1. A novel gauge-invariant by construction method in the Casimir effect.
2. The Casimir-Polder potential of an anisotropic atom between two dielectric half spaces with Chern-Simons boundary layers.
3. The Casimir-Polder potential of an anisotropic atom between two Chern-Simons layers in vacuum expressed through special functions.
4. P-odd three-body vacuum effects.
5. Casimir energy of two Chern-Simons layers in vacuum.
6. Casimir energy of two dielectric half spaces with Chern-Simons boundary layers.
7. Appearance of a minimum in the Casimir energy due to presence of Chern-Simons layers at the boundaries of dielectrics.

Chern-Simons Casimir effect

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V. N. Marachevsky, *Chern-Simons boundary layers in the Casimir effect*, Mod.Phys.Lett.A **35**, 2040015 (2020).

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Chern-Simons layer on a dielectric half space

Chern-Simons layer on a dielectric half space

The action with Chern-Simons layer at $z = 0$ has the form:

$$S = \frac{a}{2} \int \varepsilon^{z\nu\rho\sigma} A_\nu F_{\rho\sigma} dtdxdy. \tag{1}$$

Equations of electromagnetic field in the presence of Chern-Simons action (1) can be written as follows:

$$\partial_\mu F^{\mu\nu} + a\, \varepsilon^{z\nu\rho\sigma} F_{\rho\sigma} \delta(z) = 0. \tag{2}$$

Consider a flat Chern-Simons layer put at $z = 0$ on a dielectric half space $z < 0$ characterized by a frequency dependent dielectric permittivity $\varepsilon(\omega)$, the magnetic permeability $\mu = 1$. Boundary conditions on the components of the electromagnetic field follow:

$$E_z|_{z=0^+} - \varepsilon(\omega) E_z|_{z=0^-} = -2aH_z|_{z=0}, \tag{3}$$

$$H_x|_{z=0^+} - H_x|_{z=0^-} = 2aE_x|_{z=0}, \tag{4}$$

$$H_y|_{z=0^+} - H_y|_{z=0^-} = 2aE_y|_{z=0}. \tag{5}$$

A special case: plane Chern-Simons layer in vacuum

TE or s-polarization (the factor $\exp(i\omega t + ik_y y)$ is omitted):

$$E_x = \exp(-ik_z z) + r_s \exp(ik_z z), z > 0 \quad (6)$$

$$E_x = t_s \exp(-ik_z z), z < 0 \quad (7)$$

$$H_x = r_{s\to p} \exp(ik_z z), z > 0 \quad (8)$$

$$H_x = t_{s\to p} \exp(-ik_z z), z < 0. \quad (9)$$

TM or p-polarization:

$$H_x = \exp(-ik_z z) + r_p \exp(ik_z z), z > 0 \quad (10)$$

$$H_x = t_p \exp(-ik_z z), z < 0 \quad (11)$$

$$E_x = r_{p\to s} \exp(ik_z z), z > 0 \quad (12)$$

$$E_x = t_{p\to s} \exp(-ik_z z), z < 0. \quad (13)$$

A special case: plane Chern-Simons layer in vacuum

In vacuum the reflection coefficients for TE mode from a Chern-Simons layer have the form:

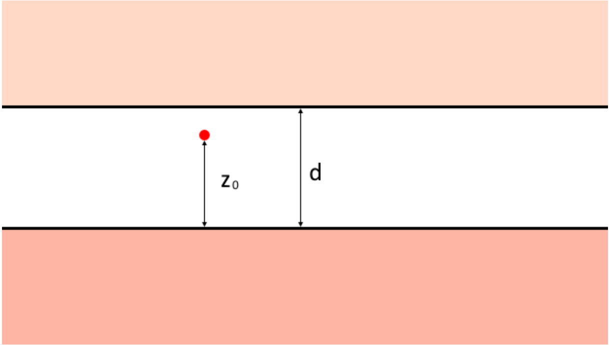
$$r_s = -\frac{a^2}{1+a^2}, \qquad t_s = \frac{1}{1+a^2},$$
$$r_{s\to p} = \frac{a}{1+a^2}, \qquad t_{s\to p} = -\frac{a}{1+a^2}, \tag{14}$$

for TM mode:

$$r_p = \frac{a^2}{1+a^2}, \qquad t_p = \frac{1}{1+a^2},$$
$$r_{p\to s} = \frac{a}{1+a^2}, \qquad t_{p\to s} = \frac{a}{1+a^2}. \tag{15}$$

[V.N.Marachevsky, Theor.Math.Phys., 2017]

The Casimir-Polder potential of an anisotropic atom between two Chern-Simons boundary layers



Anisotropic neutral atom between two dielectric half spaces with plane Chern-Simons boundary layers, z_0 is a distance of the atom from the layer and the dielectric medium characterized by the index 2, d is a width of the vacuum slit.
[V.N.Marachevsky and A.A.Sidelnikov, Phys.Rev.D, 2023].

Consider a dipole source at the point $\mathbf{r}' = (0, 0, z_0)$ characterized by electric dipole moment $d^l(t)$ with components of the four-current density [V.N.Marachevsky and Yu.M.Pis'mak, Phys.Rev.D, 2010]

$$\rho(t, \mathbf{r}) = -d^l(t)\partial_l \delta^3(\mathbf{r} - \mathbf{r}')\,, \tag{16}$$

$$j^l(t, \mathbf{r}) = \partial_t d^l(t)\delta^3(\mathbf{r} - \mathbf{r}')\,. \tag{17}$$

The Casimir-Polder potential is defined in terms of the scattered electric Green function $D_{ij}^{E,sc}(t_1 - t_2, \mathbf{r}, \mathbf{r}') = D_{ij}^{E}(t_1 - t_2, \mathbf{r}, \mathbf{r}') - D_{ij}^{E,vac}(t_1 - t_2, \mathbf{r}, \mathbf{r}')$ from the source (16),(17) and the atomic polarizability $\alpha_{ij}(t_1 - t_2) = i\langle T(\hat{d}_i(t_1), \hat{d}_j(t_2))\rangle$ as follows:

$$U(z_0) = -\int_0^\infty \frac{d\omega}{2\pi} \alpha^{ij}(i\omega) D_{ij}^{E,sc}(i\omega, \mathbf{r}', \mathbf{r}'). \tag{18}$$

From Weyl formula

$$\frac{e^{i\omega|\mathbf{r}'-\mathbf{r}|}}{4\pi|\mathbf{r}'-\mathbf{r}|} = i\iint \frac{e^{i\left(k_x(x'-x)+k_y(y'-y)+\sqrt{\omega^2-k_x^2-k_y^2}(z'-z)\right)}}{2\sqrt{\omega^2-k_x^2-k_y^2}}\frac{dk_x dk_y}{(2\pi)^2}\,, \tag{19}$$

valid for $z'-z>0$, one can write electric and magnetic fields propagating downwards from the dipole source (16),(17) in the form [V.N.Marachevsky and A.A.Sidelnikov, Universe, 2021]

$$\mathbf{E^0}(\omega,\mathbf{r}) = \int \widetilde{\mathbf{N}}(\omega,\mathbf{k}_{\|})e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}}e^{-ik_z(z-z_0)}d^2\mathbf{k}_{\|}, \tag{20}$$

$$\mathbf{H^0}(\omega,\mathbf{r}) = \frac{1}{\omega}\int [\widetilde{\mathbf{k}}\times\widetilde{\mathbf{N}}(\omega,\mathbf{k}_{\|})]e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}}e^{-ik_z(z-z_0)}d^2\mathbf{k}_{\|}, \tag{21}$$

$$\widetilde{\mathbf{N}}(\omega,\mathbf{k}_{\|}) = \frac{i}{8\pi^2 k_z}\left(-(\mathbf{d}\cdot\widetilde{\mathbf{k}})\widetilde{\mathbf{k}}+\omega^2\mathbf{d}\right), \tag{22}$$

where $\mathbf{k}_{\|}=(k_x,k_y)$, $k_z=\sqrt{\omega^2-k_{\|}^2}$, $\widetilde{\mathbf{k}}=(\mathbf{k}_{\|},-k_z)$.

To solve a diffraction problem we write electric and magnetic fields for $z > 0$ in the form

$$\mathbf{E}^1(\omega, \mathbf{r}) = \int \widetilde{\mathbf{N}}(\omega, \mathbf{k}_{||}) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-ik_z(z-z_0)} d^2\mathbf{k}_{||} + \int \mathbf{v}(\omega, \mathbf{k}_{||}) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{ik_z z} d^2\mathbf{k}_{||} \,, \tag{23}$$

$$\mathbf{H}^1(\omega, \mathbf{r}) = \frac{1}{\omega}\int [\widetilde{\mathbf{k}} \times \widetilde{\mathbf{N}}(\omega, \mathbf{k}_{||})] e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-ik_z(z-z_0)} d^2\mathbf{k}_{||} + \frac{1}{\omega}\int [\mathbf{k} \times \mathbf{v}(\omega, \mathbf{k}_{||})] e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{ik_z z} d^2\mathbf{k}_{||} \,, \tag{24}$$

and for $z < 0$ in the form

$$\mathbf{E}^2(\omega, \mathbf{r}) = \int \mathbf{u}(\omega, \mathbf{k}_{||}) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-iK_z z} d^2\mathbf{k}_{||} \,, \tag{25}$$

$$\mathbf{H}^2(\omega, \mathbf{r}) = \frac{1}{\omega}\int \left([\mathbf{k}_{||} \times \mathbf{u}(\omega, \mathbf{k}_{||})] - K_z[\mathbf{n} \times \mathbf{u}(\omega, \mathbf{k}_{||})]\right) e^{i\mathbf{k}_{\|}\cdot\mathbf{r}_{\|}} e^{-iK_z z} d^2\mathbf{k}_{||} \tag{26}$$

with $K_z = \sqrt{\varepsilon(\omega)\omega^2 - k_x^2 - k_y^2}$ and $\mathbf{n} = (0, 0, 1)$.

Unknown vector functions $\mathbf{v}(\omega, \mathbf{k}_{\|})$ and $\mathbf{u}(\omega, \mathbf{k}_{\|})$ can be found from the system of boundary conditions imposed on electric and magnetic fields:

$$\mathrm{div}(\mathbf{E^1} - \mathbf{E^0}) = 0, \tag{27}$$

$$\mathrm{div}\mathbf{E^2} = 0, \tag{28}$$

$$E_x^1|_{z=0} = E_x^2|_{z=0}, \tag{29}$$

$$E_y^1|_{z=0} = E_y^2|_{z=0}, \tag{30}$$

$$H_x^1|_{z=0+} - H_x^2|_{z=0-} = 2aE_x^1|_{z=0}, \tag{31}$$

$$H_y^1|_{z=0+} - H_y^2|_{z=0-} = 2aE_y^1|_{z=0}. \tag{32}$$

We get in polar coordinates:

$$v_r = \left[-\frac{r_{TM} + a^2 T}{1 + a^2 T} \widetilde{N}_r + \frac{k_z}{\omega} \frac{aT}{1 + a^2 T} \widetilde{N}_\theta \right] e^{ik_z z_0}, \tag{33}$$

$$v_\theta = \left[-\frac{\omega}{k_z} \frac{aT}{1 + a^2 T} \widetilde{N}_r + \frac{r_{TE} - a^2 T}{1 + a^2 T} \widetilde{N}_\theta \right] e^{ik_z z_0}, \tag{34}$$

$$v_z = \frac{k_r}{k_z} \left[\frac{r_{TM} + a^2 T}{1 + a^2 T} \widetilde{N}_r - \frac{k_z}{\omega} \frac{aT}{1 + a^2 T} \widetilde{N}_\theta \right] e^{ik_z z_0}, \tag{35}$$

where r_{TM}, r_{TE} are Fresnel reflection coefficients

$$r_{TM}(\omega, k_r) = \frac{\varepsilon(\omega) k_z - K_z}{\varepsilon(\omega) k_z + K_z}, \qquad r_{TE}(\omega, k_r) = \frac{k_z - K_z}{k_z + K_z} \tag{36}$$

and

$$T(\omega, k_r) = \frac{4 k_z K_z}{(k_z + K_z)(\varepsilon(\omega) k_z + K_z)}. \tag{37}$$

At this point it is convenient to define the local matrix R resulting from equations (33), (34):

$$R(a, \varepsilon(\omega), \omega, k_r) \equiv \frac{1}{1 + a^2 T} \begin{pmatrix} -r_{TM} - a^2 T & \frac{k_z}{\omega} a T \\ -\frac{\omega}{k_z} a T & r_{TE} - a^2 T \end{pmatrix}. \quad (38)$$

The tangential local components of the electric field in the interval $0 < z < d$ from the point dipole (16),(17) located at $(0, 0, z_0)$ are expressed in terms of matrices $R_1(\omega)$, $R_2(\omega)$ as follows:

$$\begin{pmatrix} E_r \\ E_\theta \end{pmatrix} = \frac{e^{ik_z z}}{I - R_2 R_1 e^{2ik_z d}} \left[R_2 R_1 \begin{pmatrix} N_r \\ N_\theta \end{pmatrix} e^{ik_z(2d - z_0)} + R_2 \begin{pmatrix} \widetilde{N_r} \\ \widetilde{N_\theta} \end{pmatrix} e^{ik_z z_0} \right]$$
$$+ \frac{e^{ik_z(2d - z)}}{I - R_1 R_2 e^{2ik_z d}} \left[R_1 R_2 \begin{pmatrix} \widetilde{N_r} \\ \widetilde{N_\theta} \end{pmatrix} e^{ik_z z_0} + R_1 \begin{pmatrix} N_r \\ N_\theta \end{pmatrix} e^{-ik_z z_0} \right], \quad (39)$$

in (39) the local components of the electric field are obtained by a summation of multiple reflections from media with indices 1 and 2.

It is convenient to define four matrices entering (39) after Wick rotation:

$$M^1 = \left(I - R_2(i\omega)R_1(i\omega)e^{-2k_z d}\right)^{-1} R_2(i\omega)R_1(i\omega), \tag{40}$$

$$M^2 = \left(I - R_2(i\omega)R_1(i\omega)e^{-2k_z d}\right)^{-1} R_2(i\omega), \tag{41}$$

$$M^3 = \left(I - R_1(i\omega)R_2(i\omega)e^{-2k_z d}\right)^{-1} R_1(i\omega)R_2(i\omega), \tag{42}$$

$$M^4 = \left(I - R_1(i\omega)R_2(i\omega)e^{-2k_z d}\right)^{-1} R_1(i\omega). \tag{43}$$

After integration over polar coordinates we express scattered electric Green functions at imaginary frequencies for coinciding arguments $\mathbf{r} = \mathbf{r}'$ in terms of matrix elements of matrices M:

$$D^{E,sc}_{xx}(i\omega, \mathbf{r} = \mathbf{r}') = D^{E,sc}_{yy}(i\omega, \mathbf{r} = \mathbf{r}') = -\frac{1}{8\pi}\int_0^\infty dk_r k_r$$
$$\times \Bigg[k_z(e^{-2k_z d}M^1_{11} + e^{-2k_z z_0}M^2_{11} + e^{-2k_z d}M^3_{11} + e^{-2k_z(d-z_0)}M^4_{11})$$
$$+\frac{\omega^2}{k_z}(e^{-2k_z d}M^1_{22} + e^{-2k_z z_0}M^2_{22} + e^{-2k_z d}M^3_{22} + e^{-2k_z(d-z_0)}M^4_{22})\Bigg] \tag{44}$$

$$D_{zz}^{E,sc}(i\omega, \mathbf{r}=\mathbf{r}') = -\frac{1}{4\pi}\int\limits_0^\infty dk_r \frac{k_r^3}{k_z}$$
$$\times\left[-e^{-2k_z d}M_{11}^1 + e^{-2k_z z_0}M_{11}^2 - e^{-2k_z d}M_{11}^3 + e^{-2k_z(d-z_0)}M_{11}^4)\right] \tag{45}$$

The Casimir-Polder potential can be evaluated by substituting (44), (45) into the formula

$$U(z_0) = -\int\limits_0^\infty \frac{d\omega}{2\pi}\alpha^{ij}(i\omega)D_{ij}^{E,sc}(i\omega, \mathbf{r}', \mathbf{r}'). \tag{46}$$

For Chern-Simons layers in vacuum $\varepsilon(\omega) = 1$ for $z < 0$ and $z > d$.

$$M^1 = M^3 = -\frac{1}{(1+a_1^2)(1+a_2^2)\,\det[I - R_1 R_2 e^{-2k_z d}]} \times \begin{pmatrix} a_1 a_2(1 - a_1 a_2(1 - e^{-2k_z d})) & a_1 a_2(a_1 + a_2)\frac{k_z}{\omega} \\ -a_1 a_2(a_1 + a_2)\frac{\omega}{k_z} & a_1 a_2(1 - a_1 a_2(1 - e^{-2k_z d})) \end{pmatrix}, \tag{47}$$

$$M^2 = -\frac{1}{(1+a_1^2)(1+a_2^2)\,\det[I - R_1 R_2 e^{-2k_z d}]} \times \begin{pmatrix} a_2^2(1 + a_1^2(1 - e^{-2k_z d})) & -a_2(1 + a_1^2 + a_1 a_2 e^{-2k_z d})\frac{k_z}{\omega} \\ a_2(1 + a_1^2 + a_1 a_2 e^{-2k_z d})\frac{\omega}{k_z} & a_2^2(1 + a_1^2(1 - e^{-2k_z d})) \end{pmatrix}, \tag{48}$$

$$M^4 = -\frac{1}{(1+a_1^2)(1+a_2^2)\,\det[I - R_1 R_2 e^{-2k_z d}]} \times \begin{pmatrix} a_1^2(1 + a_2^2(1 - e^{-2k_z d})) & -a_1(1 + a_2^2 + a_1 a_2 e^{-2k_z d})\frac{k_z}{\omega} \\ a_1(1 + a_2^2 + a_1 a_2 e^{-2k_z d})\frac{\omega}{k_z} & a_1^2(1 + a_2^2(1 - e^{-2k_z d})) \end{pmatrix}. \tag{49}$$

Note that

$$\frac{1}{(1+a_1^2)(1+a_2^2)\det[I - R_1 R_2 e^{-2k_z d}]}$$
$$= \frac{1}{1+a_1^2+a_2^2+2a_1a_2e^{-2k_z d}+a_1^2a_2^2(1-e^{-2k_z d})^2}$$
$$= \frac{\gamma_1}{1+\beta_1 y} + \frac{\gamma_2}{1+\beta_2 y} \quad (50)$$

with $y = \exp(-2k_z d)$, $A = a_1^2 a_2^2$, $B = 2(a_1a_2 - a_1^2a_2^2)$, $C = (1 + a_1^2)(1 + a_2^2)$, $y_{1,2} = \frac{-B\pm\sqrt{B^2-4AC}}{2A} = (a_1a_2 - 1 \pm i(a_1 + a_2))/(a_1a_2)$, $\beta_1 = -1/y_1$, $\beta_2 = -1/y_2$, $\gamma_1 = 1/(Ay_1(y_2-y_1))$, $\gamma_2 = 1/(Ay_2(y_1 - y_2))$.

Decomposition of the denominator in (50) into two terms leads to an analytic result for the Casimir-Polder potential in terms of Lerch transcendent functions. We change variables

$$\int_0^\infty k_r dk_r f(k_z) = \int_\omega^\infty k_z dk_z f(k_z) \tag{51}$$

and use the integral

$$G_0(\chi,\beta,\omega) \equiv \int_\omega^\infty \frac{e^{-2k_z\chi}}{1+\beta e^{-2k_z d}} dk_z = \frac{1}{2d} \int_0^{e^{-2\omega d}} \frac{y^{\frac{\chi}{d}-1}}{1+\beta y} dy$$
$$= \frac{e^{-2\omega\chi}}{2d} \Phi\left(-\beta e^{-2\omega d}, 1, \frac{\chi}{d}\right), \tag{52}$$

where $\Phi(\alpha_1, \alpha_2, \alpha_3)$ is a Lerch transcendent function.

At large distances of the atom from half spaces the Casimir-Polder potential has the following form:

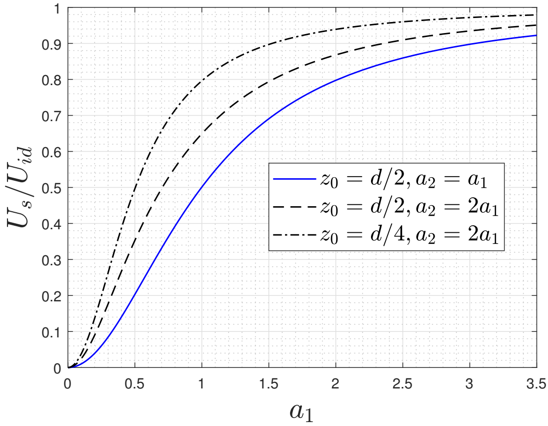
$$U_s(z_0, d) = U_{s1}(z_0, d) + U_{s2}(d) = \frac{\alpha_{xx}(0) + \alpha_{yy}(0) + \alpha_{zz}(0)}{32\pi^2 d^4}$$

$$\times \sum_{i=1,2} \gamma_i \Big[-a_2^2(1+a_1^2)\Phi\Big(y_i^{-1}, 4, \frac{z_0}{d}\Big) - a_1^2(1+a_2^2)\Phi\Big(y_i^{-1}, 4, \frac{d-z_0}{d}\Big)$$

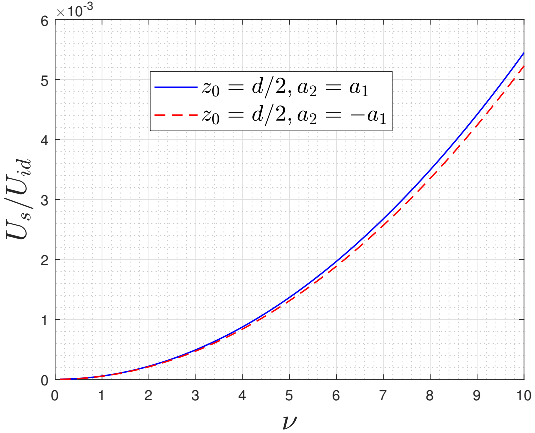
$$+ a_1^2 a_2^2 \Phi\Big(y_i^{-1}, 4, \frac{d+z_0}{d}\Big) + a_1^2 a_2^2 \Phi\Big(y_i^{-1}, 4, \frac{2d-z_0}{d}\Big)\Big] + U_{s2}(d), \quad (53)$$

$$U_{s2}(d) = \frac{\alpha_{xx}(0) + \alpha_{yy}(0) - \alpha_{zz}(0)}{32\pi^2 d^4}\Big(\mathrm{Li}_4\Big(\frac{a_1 a_2}{(a_1+i)(a_2+i)}\Big) + \mathrm{Li}_4\Big(\frac{a_1 a_2}{(a_1-i)(a_2-i)}\Big)\Big). \quad (54)$$

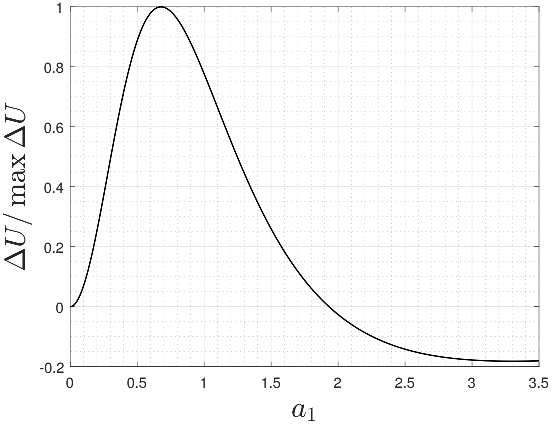
Here $\Phi(\alpha_1, \alpha_2, \alpha_3)$ - Lerch transcendent function, $\mathrm{Li}_4(z)$ is a polylogarithm function, $y_{1,2} = (a_1 a_2 - 1 \pm i(a_1 + a_2))/(a_1 a_2)$, $\gamma_1 = 1/(A y_1(y_2 - y_1))$, $\gamma_2 = 1/(A y_2(y_1 - y_2))$, $A = a_1^2 a_2^2$.



Ratios of the Casimir-Polder potential of a neutral polarizable isotropic atom located between two plane Chern-Simons layers in vacuum $U_s(z_0, d)$ to the potential of the same atom between two perfectly conducting planes $U_{id}(z_0, d)$, here z_0 is a distance of the atom from the layer characterized by a constant a_2, d is a distance between the layers.

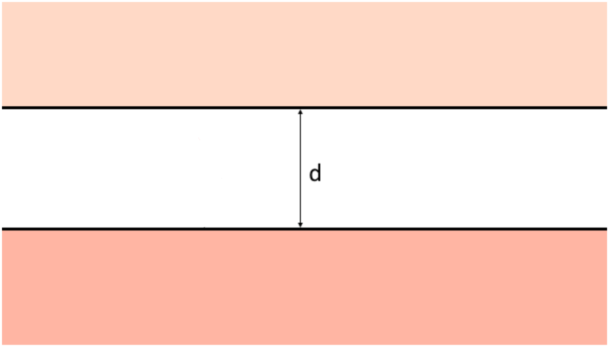


Ratios of the Casimir-Polder potentials $U_s(z_0, d)/U_{id}(z_0, d)$ differing by 180 degree rotation of the Chern-Simons layer characterized by a parameter a_2: $a_2 = a_1$ and $a_2 = -a_1$. Here z_0 is a distance of the atom from the layer characterized by a constant a_2, d is a distance between the layers, a dimensionless parameter $\nu = a_1/\alpha$ is quantized in quantum Hall layers and Chern insulators.



Ratio $\Delta U = U_s(z_0 = d/2, d, a_2 = -a_1) - U_s(z_0 = d/2, d, a_2 = a_1)$ to $\max \Delta U \approx 0.00587 |U_{id}(z_0 = d/2, d)|$, $\max \Delta U$ holds at $a_1 \approx 0.678$.

Casimir energy of two Chern-Simons layers



Chern-Simons plane layers in vacuum characterized by a_1 and a_2 are located at $z = d$ and $z = 0$ respectively. We define the matrix $R_{down} = R(a_2)$ of the reflection of electromagnetic waves (propagating downwards) from the Chern–Simons layer with $z = 0$:

$$R(a_2) = \begin{pmatrix} r_s & r_{p\to s} \\ r_{s\to p} & r_p \end{pmatrix} = \frac{a_2}{1+a_2^2}\begin{pmatrix} -a_2 & 1 \\ 1 & a_2 \end{pmatrix} \tag{55}$$

We write the matrix R_{up} of reflection of electromagnetic waves (propagating upwards) from the layer with $z = d$. After the Euclidean rotation we get

$$R_{up} = SR(a_1)S, \tag{56}$$

where

$$S = \begin{pmatrix} e^{-d\sqrt{\omega^2+k_x^2+k_y^2}} & 0 \\ 0 & e^{-d\sqrt{\omega^2+k_x^2+k_y^2}}. \end{pmatrix} \tag{57}$$

is the shift matrix that arises from the coordinate change $x_1 = x$, $y_1 = -y$, $z_1 = -z + d$.

Argument principle

$$\frac{1}{2\pi i}\oint \phi(\omega)\frac{d}{d\omega}\ln f(\omega)d\omega = \sum \phi(\omega_0) - \sum \phi(\omega_\infty) \qquad (58)$$

$$\phi(\omega) = \omega/2$$

$$f(\omega) = \det(I - R_{down}(\omega)R_{up}(\omega))$$

Casimir energy of two Chern-Simons layers in vacuum

The Casimir energy of two Chern-Simons layers in vacuum is [V.N.Marachevsky, Theor.Math.Phys., 2017]

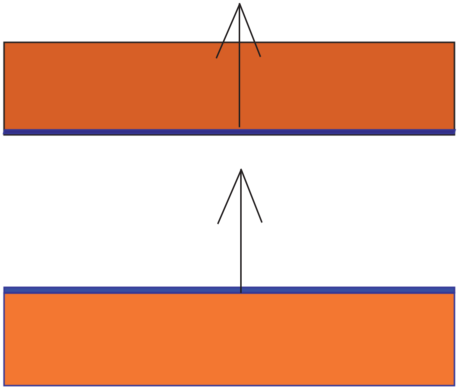
$$E(-a_1, a_2, d) = \frac{1}{2}\iiint \frac{d\omega dk_x dk_y}{(2\pi)^3} \ln\det(I - R_{up}R_{down}) = \\ \frac{1}{4\pi^2}\int_0^{+\infty} dr r^2 \ln\det(I - e^{-2dr}R(a_1)R(a_2)) = \\ \frac{1}{4\pi^2}\int_0^{+\infty} dr r^2 \ln\det(I - e^{-2dr}Q), \tag{59}$$

where

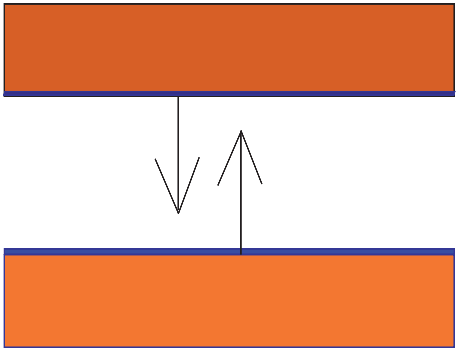
$$Q = a_1 a_2 \begin{pmatrix} \frac{1}{(a_1-i)(a_2+i)} & 0 \\ 0 & \frac{1}{(a_1+i)(a_2-i)} \end{pmatrix}. \tag{60}$$

$$E(a_1, a_2, d) = -\frac{1}{16\pi^2 d^3}\left(\mathrm{Li}_4\left(\frac{a_1 a_2}{(a_1+i)(a_2+i)}\right) + \mathrm{Li}_4\left(\frac{a_1 a_2}{(a_1-i)(a_2-i)}\right)\right), \qquad (61)$$

where $\mathrm{Li}_4(x) = \sum_{k=1}^{+\infty} x^k/k^4 = -\frac{1}{2}\int_0^{+\infty} drr^2 \ln(1 - xe^{-r})$.
Note that for $a_1 = -a_2$ the force is attractive for every a_1 (due to a theorem that the Casimir force between mirror objects is attractive). For $a_1 = a_2$ [V. N. Markov and Yu. M. Pis'mak, J. Phys. A: Math. Gen., 2006] one gets the Casimir energy of two Chern-Simons layers with identically selected directions of the layers in space. In this case the force is repulsive at all distances d for $a_1 \in [0, a_0]$, where $a_0 \approx 1.032502$, and attractive at all distances d for $a_1 > a_0$.



$a_1 = a_2$ case is shown, leads to repulsion for two layers in vacuum for $a_1 \in [0, a_0]$, where $a_0 \approx 1.032502$, and to attraction for $a_1 > a_0$.



$a_1 = -a_2$ is shown, leads to attraction in vacuum and for coinciding dielectrics.

The Casimir effect for Chern-Simons layers at the boundaries of dielectric and metal half spaces

[V.N.Marachevsky, Phys.Rev.B, 2019]
[V.N.Marachevsky, Mod.Phys.Lett.A, 2020]

Consider two dielectric half spaces with Chern-Simons terms characterized by constants a_1, a_2 on their surfaces respectively. Assume there is a vacuum slit L between half spaces.
The reflection matrix $R_{down} = R(a_2)$ from the $z \leq 0$ half space is defined by:

$$R(a_2) = \begin{pmatrix} r_s & r_{p\to s} \\ r_{s\to p} & r_p \end{pmatrix} = \frac{1}{1+a_2^2 T} \begin{pmatrix} r_s^f - a_2^2 T & a_2 T \\ a_2 T & r_p^f + a_2^2 T \end{pmatrix}. \tag{62}$$

The reflection matrix from the $z \geq L$ half space is defined after euclidean rotation by

$$R_{up} = SR(a_1)S, \tag{63}$$

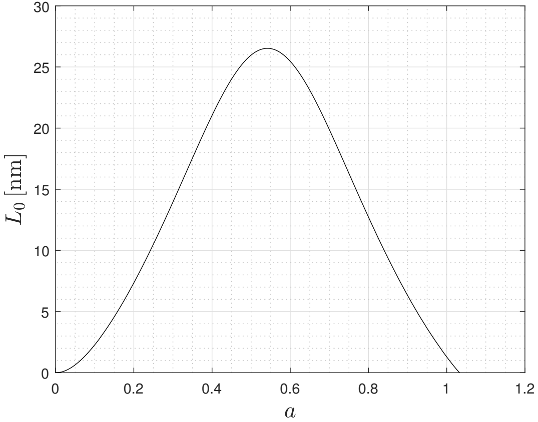
where

$$S = \begin{pmatrix} e^{-L\sqrt{\omega^2+k_x^2+k_y^2}} & 0 \\ 0 & e^{-L\sqrt{\omega^2+k_x^2+k_y^2}} \end{pmatrix} \tag{64}$$

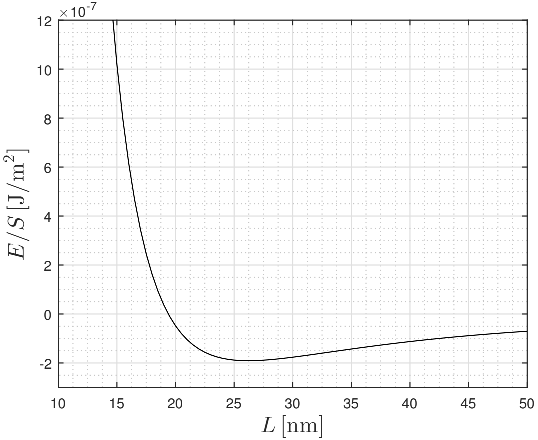
is a matrix due to a change of the coordinate system $x_1 = x, y_1 = -y, z_1 = -z + L$.

The Casimir energy is equal

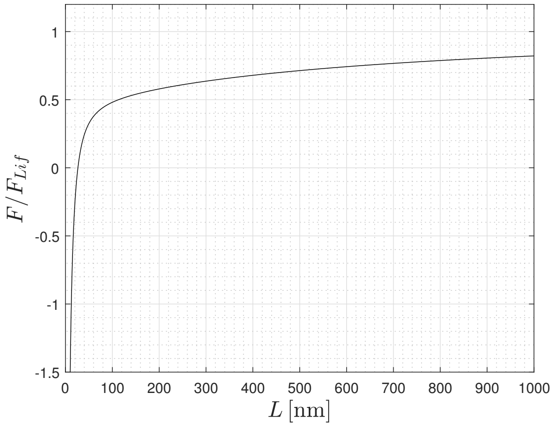
$$E(-a_1, a_2, L) = \frac{1}{2} \iiint \frac{d\omega dk_x dk_y}{(2\pi)^3} \ln \det(I - R_{up} R_{down}) = \frac{1}{4\pi^2} \int_0^{+\infty} drr^2 \ln \det(I - e^{-2Lr} R(a_1) R(a_2)). \quad (65)$$



Position of the minimum of the energy L_0 for Chern-Simons layers at the boundaries of two $\mathrm{SiO_2}$ glass half spaces, $a \equiv a_1 = a_2$.



Energy on a unit surface for Chern-Simons layers with $a_1 = a_2 = 0.542$ at the boundaries of two $\mathrm{SiO_2}$ glass half spaces. The minimum of the energy is at $L_0 = 26.52$ nm.



Ratio of the force F with Chern-Simons layers at the boundaries of two SiO_2 glass half spaces to the force F_{Lif} between two SiO_2 glass half spaces. Here $a_1 = a_2 = 0.542$.

Explaining the minimum of the Casimir energy

Lifshitz force power law between two dielectrics/metals effectively changes from retarded L^{-4} to nonretarded L^{-3} behaviour at distances of the order $L \sim 10$ nm.

On the other hand, the force between two Chern-Simons layers in vacuum has L^{-4} behavior at all separations and thus dominates the total force at separations of the order $L \lesssim 10$ nm. For the condition $a \equiv a_1 = a_2$ the Casimir force between two Chern-Simons layers in vacuum is repulsive at all distances L for an interval $a \in [0, a_0]$, where $a_0 \approx 1.032502$.

As a result, the sum of the Lifshitz force and the force between two Chern-Simons layers in vacuum effectively leads to a repulsive force at short separations and to an attractive force at large separations.

Conclusions

1. A novel gauge-invariant formalism in the Casimir effect is presented.
2. Analytic results for the Casimir-Polder potential of a neutral anisotropic atom between two half-spaces with Chern-Simons boundary layers are derived and expressed through Lerch transcendent functions and polylogarithms.
3. P-odd three-body vacuum effects are predicted: there is a difference in values of the Casimir-Polder potential of a neutral atom after 180 degree rotation of one of the Chern-Simons layers. A neutral atom is described by QED dipole interaction.

Conclusions

4. A diffraction problem for reflection of an electrmagnetic wave from a dielectric with Chern-Simons layer at its surface is solved.
5. The Casimir energy of two Chern-Simons layers and two Chern-Simons layers on top of dielectrics separated by a vacuum slit is derived in a scattering approach in terms of reflection coefficients.
6. Existence of a regime with the minimum of the Casimir energy due to presence of Chern-Simons layers at the surfaces of dielectrics at a distance of the order 10 nm, the Casimir force in this case is attractive at large distances and repulsive at short distances between the two dielectrics with Chern-Simons boundary layers.

Acknowledgments

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