Cosmological particle creation in Weyl geometry

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Weyl Geometry

The gauge invariance of the classical electrodynamics is translated into the local conformal invariance of the curvature tensor, $R^{\mu}{}_{\nu\lambda\sigma}$, Ricci tensor $R_{\mu\nu}$ and strength tensor $F_{\mu\nu}$:

$$R^{\mu}{}_{\nu\lambda\sigma} = \hat{R}^{\mu}{}_{\nu\lambda\sigma}$$

$$R_{\mu\nu} = \hat{R}_{\mu\nu}$$

$$F_{\mu\nu} = \hat{F}_{\mu\nu}.$$

Here $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, "hat", "^", means — "confomally transformed";
Local conformal transformation:

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = \Omega^2(x) d\hat{s}^2 = \Omega^2(x) \hat{g}_{\mu\nu} dx^{\mu} dx^{\nu}.$$

$\Omega(x)$ — conformal factor
We will consider A_{μ} not as an electromagnetic potential (like Hermann Weyl did), but as some part (vector one) of the Weyl geometry.

This is the "Weyl Geometry"

I. Conformal invariant gravity

Electromagnetic equations are conformal invariant (H.Weyl).

Hermann Weyl claims:

In the unified theory gravitational equations should be conformal invariant.

Quadratic gravity — why?

1. Most natural in constructing conformal invariant action integral (Lagrangian) in 4-dim
2. Appears in the trace anomaly formulas for one-loop quantum calculations:
   A.D. Sakharov's induced gravity (1966)
   Ya.B. Zel'dovich and A.A. Starobinsky
   L. Parker and S. Fulling
   A.A. Grib, S.G. Mamaev, V.M. Mostepanenko
   ...

$$S_{\mathrm{W}} = \int \mathcal{L}_{\mathrm{W}} \sqrt{-g}\, d^4x$$

$$\mathcal{L}_{\mathrm{W}} = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 F_{\mu\nu} F^{\mu\nu}$$

Its counterpart in Riemannian geometry:

$$\mathcal{L}_2 = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda \quad (A_\mu = 0)$$

can be rewritten as

$$\mathcal{L}_2 = \alpha C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma} + \beta GB + \gamma R^2 + \alpha_4 R + \alpha_5 \Lambda$$

$$\alpha = 2\alpha_1 + \frac{1}{2}\alpha_2, \quad \beta = -\alpha_1 - \frac{1}{2}\alpha_2, \quad \gamma = \frac{1}{3}(\alpha_1 + \alpha_2 + 3\alpha_3)$$

$C_{\mu\nu\lambda\sigma}$ — Weyl tensor
(completely traceless part of the curvature tensor)

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - \frac{1}{2}(R_{\mu\lambda} g_{\nu\sigma} - R_{\nu\sigma} g_{\mu\lambda} +$$

$$+ R_{\mu\sigma} g_{\nu\lambda} - R_{\nu\lambda} g_{\mu\sigma}) + \frac{1}{6} R (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda})$$

Gauss-Bonnet term

$$GB = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

In 4-dim GB — full derivative, does not affect field equations

Conformal invariance

$$C^{\mu}{}_{\nu\lambda\sigma} = \hat{C}^{\mu}{}_{\nu\lambda\sigma} \quad \Rightarrow \quad C^2\sqrt{-g} = \hat{C}^2\sqrt{-\hat{g}} \quad \Rightarrow$$

the only term in $\mathcal{L}_2$ left.

Total action

$$S_{\mathrm{tot}} = S_{\mathrm{W}} + S_{\mathrm{m}}$$

Important! The action integral for the matter fields, S_{m}, does not need to be conformal invariant, but its variation, δS_{m}, does.

$$\delta S_{\mathrm{m}} = -\frac{1}{2}\int T^{\mu\nu}(\delta g_{\mu\nu})\sqrt{-g}\, d^4x - \int G^{\mu}(\delta A_{\mu})\sqrt{-g}\, d^4x$$

$$+ \int \frac{\partial \mathcal{L}_{\mathrm{W}}}{\partial \psi}(\delta\psi)\sqrt{-g}\, d^4x = 0$$

ψ — collective dynamical variable describing the matter fields, $\frac{\delta S_{\mathrm{m}}}{\delta \psi} = 0$:

$$\frac{\delta S_{\mathrm{m}}}{\delta \Omega} = 0 \;\Rightarrow\; ?$$

Total action II

Let

$$\delta g_{\mu\nu} = 2\Omega \hat{g}_{\mu\nu}(\delta\Omega) = 2 g_{\mu\nu} \frac{\delta\Omega}{\Omega}$$

$$\delta A_\mu = 2\delta\left(\frac{\Omega_{,\mu}}{\Omega}\right) = 2\delta(\log\Omega)_{,\mu} = 2(\delta(\log\Omega))_{,\mu} = 2\left(\frac{\delta\Omega}{\Omega}\right)_{,\mu}$$

Then,

$$0 = -\int T^{\mu\nu} g_{\mu\nu} \left(\frac{\delta\Omega}{\Omega}\right) \sqrt{-g}\, d^4x - 2\int G^\mu \left(\frac{\delta\Omega}{\Omega}\right)_{,\mu} \sqrt{-g}\, d^4x \quad \Rightarrow$$

$$2G^\mu_{;\mu} = Trace T^{\mu\nu}$$

This is the self-consistency condition (consequence of field equations).

Field equations I

Vector equation (δA_μ)

$$2\alpha_1\left\{\mathcal{D}_\sigma{}^{\mu\sigma}+\mathcal{D}_\sigma{}^{\sigma\mu}-\mathcal{D}^{\mu\sigma}{}_\sigma\right\}+$$

$$\alpha_2\left\{g^{\mu\mu'}g^{\nu\nu'}\nabla_\nu(3R_{\mu'\nu'}+R_{\nu'\mu'})-g^{\alpha\beta}g^{\lambda\mu}(\nabla_\lambda 3R_{\alpha\beta})\right\}+$$

$$2\alpha_3\left\{-3g^{\mu\nu}g^{\alpha\beta}(\nabla_\nu R_{\alpha\beta})\right\}+4\alpha_4\nabla_\nu F^{\mu\nu}=G^\mu$$

Where

$$\mathcal{D}_{\mu\nu\sigma}=g_{\mu\mu'}g^{\lambda\lambda'}(\nabla_\lambda R^{\mu'}{}_{\nu\lambda'\sigma}),$$

$\nabla_\lambda I^\mu = I^\mu_\lambda + \Gamma^\mu_{\lambda\nu}I^\nu$ — covariant derivative.

$$G^\mu - ? \quad G^\mu \stackrel{\mathrm{def}}{=} -\frac{\delta S_{\mathrm{m}}}{\delta A_\mu}$$

Field equations II

Tensor equation $(\delta g_{\alpha\beta})$

$$2\alpha_1\Big\{g^{\lambda\lambda'}\big(g^{\beta\beta'}g^{\sigma\sigma'}(\nabla_\sigma\nabla_\lambda R^{\alpha}{}_{\beta'\lambda'\sigma'})+$$

$$+g^{\beta\beta'}g^{\sigma\sigma'}(\nabla_\sigma\nabla_\lambda R^{\alpha}{}_{\beta'\lambda'\sigma'}-g^{\alpha\alpha'}g^{\beta\beta'}(\nabla_\sigma\nabla_\lambda R^{\sigma}{}_{\alpha'\lambda'\beta''}))\Big\}+$$

$$+\alpha_2\Big\{g^{\alpha\alpha'}g^{\beta\beta'}g^{\nu\nu'}(\nabla_\alpha\nabla_{\alpha'}R_{\beta'\nu'})+g^{\alpha\alpha'}g^{\nu\nu'}g^{\beta\beta'}(\nabla_\nu\nabla_{\alpha'}R_{\nu'\beta'})-$$

$$-g^{\nu\nu'}g^{\alpha\alpha'}g^{\beta\beta'}(\nabla_\nu\nabla_{\nu'}R_{\alpha'\beta'})-g^{\nu\nu'}g^{\lambda\lambda'}g^{\alpha\beta}(\nabla_\nu\nabla_\lambda R_{\nu'\lambda'})\Big\}+$$

$$+2\alpha_3\Big\{(g^{\alpha\alpha'}g^{\beta\beta'}-g^{\alpha\beta}g^{\alpha'\beta'})g^{\mu\mu'}(\nabla_{\alpha'}\nabla_{\beta'}R_{\mu\mu'})\Big\}-$$

$$-2\Big\{\alpha_1 R^{\mu\nu\lambda\alpha}R_{\mu\nu\lambda}{}^{\beta}-\alpha_2(R^{\alpha}_{\ \nu}R^{\beta\nu}+2R^{\beta\mu}F_{\mu}{}^{\alpha}+2F^{\alpha\mu}F_{\mu}{}^{\beta})+$$

$$+\frac{1}{2}\alpha_3 R(R^{\alpha\beta}+R^{\beta\alpha})+\alpha_4 F^{\alpha}_{\ \nu}F^{\beta\nu}-\frac{1}{4}g^{\alpha\beta}\mathcal{L}_{\mathrm{W}}\Big\}=\frac{1}{2}T^{\alpha\beta}$$

II. Perfect fluid

Perfect fluid in Riemannian geometry (J.R. Ray, 1972):

$$S_{\rm m} = -\int \varepsilon(X,n)\sqrt{-g}\,d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\,d^4x$$

$$+ \int \lambda_1 (nu^\mu)_{;\mu}\sqrt{-g}\,d^4x + \int \lambda_2 X_{,\mu}u^\mu \sqrt{-g}\,d^4x$$

Dynamical variables:
$n(x)$ — invariant particle number density,
$u^\mu(x)$ — four-velocity vector,
$X(x)$ — auxiliary variable,
$\lambda_i(x)$ — Lagrange multipliers,
$\varepsilon(X,n)$ — invariant energy density

Equations of motion

$$\delta n: \quad -\frac{\partial \varepsilon}{\partial n} - \lambda_{1,\mu} u^\mu = 0$$

$$\delta u^\mu: \quad 2\lambda_0 u^\mu - n\lambda_{1,\mu} + \lambda_2 X_{,\mu} = 0$$

$$\delta X: \quad -\frac{\partial \varepsilon}{\partial X} - (\lambda_2 u^\mu)_{;\mu} = 0$$

Constraints:

$\delta\lambda_0$: $u^\mu u_\mu - 1 = 0$ — four-velocity

$\delta\lambda_1$: $(nu^\mu)_{;\mu} = 0$ — particle number conservation

$\delta\lambda_2$: $X_{,\mu} u^\mu = 0$ — trajectory numbering

Energy-momentum tensor

$$\delta g_{\mu\nu}: \quad T^{\mu\nu} = \varepsilon g^{\mu\nu} - 2\lambda_0 u^\mu u^\nu + n\lambda_{1,\sigma} u^\sigma g^{\mu\nu}$$

$$2\lambda_0 = -n\frac{\partial\varepsilon}{\partial n}$$

Hydrodynamical pressure p:

$$p = n\frac{\partial\varepsilon}{\partial n} - \varepsilon \quad \Rightarrow$$

$$(\varepsilon + p)u_\mu + n\lambda_{1,\mu} - \lambda_2 X_{,\mu} \quad \Rightarrow \quad \text{Euler equation}$$

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}$$

Perfect fluid $\rightarrow$ dust $\rightarrow$ single particle

Single particle in the given gravitational field

Dynamical variable: $x^\mu(\tau)$ – particle trajectory, τ – proper time

Riemannian geometry

The only invariant — the interval, s, along the trajectory $\Rightarrow$ (everybody knows)

$$S_{\rm part} = -m\int ds = -m\int\sqrt{g^{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,d\tau$$

m — particle mass

$\delta S_{\rm part} = 0 \;\Rightarrow$ "shortest" interval $=$ geodesics

$$\frac{d}{d\tau}\left(g_{\mu\lambda}(x)\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}g_{\mu\nu,\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$\frac{dx^\mu}{d\tau} = u^\mu$ — four-velocity vector $\Rightarrow$

$$u_{\lambda,\nu}u^\nu - \frac{1}{2}g_{\mu\nu,\lambda}u^\mu u^\nu = 0 \quad \Rightarrow$$

$$u_{\lambda;\nu}u^\nu = 0 \quad = \text{ geodesics}$$

$$(u_{\lambda;\nu}u^\nu u^\lambda \equiv 0)$$

Weyl geometry

$A_\mu \quad \rightarrow$ yet another invariant (!)

$$B = A_\mu u^\mu \quad \Rightarrow$$

$$S_{\rm part} = \int f_1(B) ds + \int f_2(B) d\tau = \int \{ f_1(B) \sqrt{g_{\mu\nu} u^\mu u^\nu} + f_2(B) \} d\tau$$

Equations of motion

$$f_1 u_{\lambda;\mu} u^\mu = \left((f_1'' + f_1'') A_\lambda - f_1' u_\lambda \right) B_{,\mu} u^\mu + (f_1' + f_2') F_{\lambda\mu} u^\mu$$

$$F_{\lambda\mu} = A_{\mu,\lambda} - A_{\lambda,\mu}. \quad F_{\lambda\mu} u^\lambda u^\mu \equiv 0$$

$$u_{\lambda;\sigma} u^\lambda \equiv 0 \quad \Rightarrow$$

Either

$$(f_1'' + f_1'') B - f_1' = 0$$

or

$$B_{,\mu} u^\mu = 0$$

Perfect fluid in Weyl geometry

How to insert the interaction with the Weyl vector $= A^{\mu}$ into the perfect fluid Lagrangian?

$$S_{\mathrm{m}} = -\int \varepsilon(X, n)\sqrt{-g}\, d^4x + \int \lambda_0(u_{\mu}u^{\mu} - 1)\sqrt{-g}\, d^4x$$

Evidently, the new invariant $B_{,\mu}u^{\mu}$ is tightly linked to the particle number density n. Hence it is natural to make the replacement

$$n \;\; \rightarrow \;\; \varphi(B)n \;\Rightarrow\; \varepsilon = \varepsilon(X, Z), \;\; Z = \varphi(B)n(x)$$

And how about the particle number conservation law, $(nu^{\mu})_{;\mu} = 0$?

Lyrical digression:

$$\int\limits_{\Omega} (nu^{\mu})_{;\mu}\sqrt{-g}\, d^4x = \int\limits_{\Omega} (nu^{\mu})_{,\mu} d^4x + \int\limits_{\Sigma} nu^{\mu}\sqrt{-g}\, dS_{\mu}$$

Let $\Sigma = \Sigma_{\mathrm{in}} + \Sigma_{\mathrm{out}} +$ reflecting walls,
$\Sigma_{\mathrm{in}}: \;\; t = t_{\mathrm{in}} = const$, $\Sigma_{\mathrm{out}}: \;\; t = t_{\mathrm{out}} = const$, then

Perfect fluid in Weyl geometry II

$$\int_{\Sigma} n u^{\mu} \sqrt{-g}\, dS_{\mu} = \int_{t=t_{\rm out}} n\sqrt{\gamma}\, d^3x - \int_{t=t_{\rm in}} n\sqrt{\gamma}\, d^3x = N_{\rm out} - N_{\rm in}$$

$$(n u^{\mu})_{;\mu} = 0 \quad \Rightarrow \quad N_{\rm out} = N_{\rm in} \quad \Rightarrow$$

No particle creation in between!
Quadratic gravity $\rightarrow$ conformal anomaly in quadratic field theory $\rightarrow$ vacuum polarization.
Physical vacuum is not absolutely empty, it is filled with the virtual particles, some of them are ready to become the real ones $\rightarrow$ particle creation. In the absence of the classical fields (electromagnetic, scalar,...) they are created from the vacuum itself. i. e., by the geometry.

$$(n u^{\mu})_{;\mu} = 0 \quad \Rightarrow \quad (n u^{\mu})_{;\mu} = \Phi(\text{inv}).$$

Φ — ?

Conformal transformation

$$n = \frac{\hat{n}}{\Omega^3}, \quad u^\mu = \frac{\hat{u}^\mu}{\Omega}, \quad \sqrt{-g} = \Omega^4 \sqrt{-\hat{g}} \ \Rightarrow$$

$$(nu^\mu)_{;\mu}\sqrt{-g} = (nu^\mu)_{,\mu} = (\hat{n}\hat{u}^\mu)_{,\mu}\sqrt{-\hat{g}}$$

Conformal invariance ! $\Rightarrow$

$$\Phi = \alpha_1' R^{\mu\nu\lambda\sigma}_{\mu\nu\lambda\sigma} + \alpha_2' R^{\mu\nu}_{\mu\nu} + \alpha_3' R^2 + \alpha_4' F_{\mu\nu} F^{\mu\nu}$$

$\Phi\sqrt{-g}$ is conformal invariant.

The matter action integral $S_{\rm m}$ becomes now

$$S_{\rm m} = -\int \varepsilon(X, \varphi(B)n)\sqrt{-g}\, d^4x + \int \lambda_0(u_\mu u^\mu - 1)\sqrt{-g}\, d^4x$$

$$+ \int \lambda_1 \left((\varphi_1(B) n u^\mu)_{;\mu} - \Phi(B, n)\right)\sqrt{-g}\, d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g}\, d^4x$$

and equations of motion take the form

Equations of motion I

$$\begin{cases} (\varepsilon + p)\frac{\varphi'(B)}{\varphi(B)}(Bu_\mu - A_\mu) - (\varepsilon + p)u_\mu - n\lambda_{1,\mu} + \lambda_2 X = 0 \\ \\ -\frac{\partial\varepsilon}{\partial X} - (\lambda_2 u^\mu)_{;\mu} = 0 \end{cases}$$

with the constraints

$$\begin{cases} u_\mu u^\mu - 1 = 0 \\ (nu^\mu)_{;\mu} - \Phi(\mathrm{inv}) = 0 \\ X_\mu u^\mu = 0 \end{cases}$$

Besides, there appeared the current vector $G^\mu \stackrel{\mathrm{def}}{=} -\delta S_\mathrm{m}/\delta A_\mu$. It consists of contributions from both energy density, ε — $G^\mu[\mathrm{part}]$, and creation law, $\Phi(\mathrm{inv})$ — $G^\mu[\mathrm{cr}]$. The former one is easily calculated,

$$G^\mu[\mathrm{part}] = \frac{\varphi'}{\varphi}(\varepsilon + \mathrm{p})\mathrm{u}^\mu.$$

Equations of motion II

while the latter requires some time and efforts. The Result is

$$G^{\mu}[\mathrm{cr}] = -2\alpha_1' \mathrm{g}^{\mu\sigma}\mathrm{g}^{\kappa\delta}\{2\nabla_\kappa(\lambda_1 \mathrm{R}_{\sigma\delta}) - \nabla_\kappa(\lambda_1 \mathrm{F}_{\sigma\delta})\}$$

$$-\alpha_2' g^{\mu\sigma} g^{\kappa\delta}\{2\nabla_\kappa(\lambda_1 R_{\sigma\delta}) + \nabla_\sigma(\lambda_1 R_{\kappa\delta})\}$$

$$-6\alpha_3' g^{\mu\sigma} g^{\kappa\delta}\nabla_\sigma(\lambda_1 R_{\kappa\delta}) + 4\alpha_4'(\lambda_1 F^{\nu\mu})_{;\nu}$$

The energy momentum tensor is also changed. We will write down here only $T^{\mu\nu}[\mathrm{part}]$:

$$F^{\mu\nu}[\mathrm{part}] = (\varepsilon + \mathrm{p})\left(1 - \mathrm{B}\frac{\varphi'}{\varphi}\right)\mathrm{u}^\mu \mathrm{u}^\nu - \mathrm{p}\mathrm{g}^{\mu\nu}$$

The creation part, $T^{\mu\nu}[\mathrm{cr}]$, is too lengthy and non-observable. We will show it in the next Section, for cosmological space-times.

Cosmology = homogeneity and isotropy $\Rightarrow$

Robertson-Walker metric — what is it?

3-dim space

Euclidean space (flat) $\Rightarrow$

$$dl^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$r^2 = x^2 + y^2 + z^2$ — center — at any point

Curved Space $\Rightarrow$ center at any point (homogeneity), spherical coordinates (isotropy)

$$dl^2 = f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$r \stackrel{\text{def}}{=}$ radius of a sphere

Let us assume 3-dim geometry is Riemannian (to be justified below) $\Rightarrow$
3-dim curvature scalar, K, must be constant (homogeneity)

$$K = K_0$$

$$\frac{2f'}{rf^2} + \frac{2}{r^2}\left(1 - \frac{1}{f}\right) = K_0 = const \quad \Rightarrow$$

$$\frac{1}{f} = 1 - \frac{K_0}{6}r^2 + \frac{K_1}{r}$$

$r = 0$ — singular point. Homogeneity $\rightarrow$ no singular points $\rightarrow$ $K_1 = 0$. Rescaling r $(K_0 \neq 0)$ $\Rightarrow$

$$dl^2 = a_0^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right), \quad k = 0, \pm 1$$

4-dim metric: unique cosmological time (homogeneity)

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

$a(t)$ — scale factor

Riemannian geometry: Quadratic gravity

$$\mathcal{L}_2 = \alpha C^2 + \gamma R^2 + \alpha_4 R + \alpha_5 \Lambda$$

For any scale factor $a(t)$ the Weyl tensor $C_{\mu\nu\lambda\sigma}$ is identically zero.

We are left with the Starobinsky model.

Conformal invariance $\Rightarrow$

There are no non-vacuum ($T^{\mu}_{\nu} \neq= 0$) cosmological solutions
in the conformally invariant Riemannian gravitational theory!

Weyl geometry

Why conformal invariant cosmology?
Creation of the universe from "nothing"
(quantum tunneling)
A.A. Friedmann (1923)
A.V. Vilenkin (1984)
Ya.B. Zel'dovich
S.W. Hawking

Creation probability

$$P \sim e^{-S_{\rm tot}}$$

$S_{\rm tot}$ — total action integral under the potential barrier,
$S_{\rm tot} = S_{\rm grav} + S_{\rm matter}$
Universe is created being empty $\rightarrow$ $S_{\rm matter} = 0$ $\Rightarrow$
The smaller $S_{\rm grav}$, the better.
The more symmetry, the smaller $S_{\rm grav}$.
G.'tHooft, R.Penrose — the fundamental principle of Nature

Homogeneity and isotropy

$$a(t),\ A_\mu(x) = (A(t), 0, 0, 0) \Rightarrow A_0(t) \Rightarrow F_{\mu\nu} = 0$$

$$T_\mu^\nu = (T_0^0, T_1^1 = T_2^2 = T_3^3) = T_\mu^\nu(t)$$

Gauge fixing

1. $ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) =$
   $= dt^2 - a^2(t)\gamma_{ij}dx^i dx^j = a^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j) =$
   $= \Omega^2(\eta)\hat{a}^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j)$, η — "conformal time"
   Gauge fixing $\Rightarrow$ $\hat{a}(\eta) = 1$.
   Not very interesting: how to compare with the observations?
   (the cosmological time t depends on he choice of $\Omega(\eta)$)
2. $A_0(t) = A(t) = \hat{A}(t) + \frac{2\dot{\Omega}}{\Omega}$
   Gauge fixing $\Rightarrow$ $\hat{A}(t) = 0!$ $\Rightarrow$ $B = 0$
   Advantage: all the functions of B are converted into the set of some constants. The solution in such a gauge we call "basic solution". For brevity, in what follows we will omit the "hats".

Cosmological field equations

We are not allowed to put $A = 0$ straight in the Lagrangian because $\delta A \neq 0$
Hence, one should extract the δA_μ-variation:

$$G^\mu[\mathrm{cr}] = -4\alpha_1'(\lambda_1 \mathrm{R}_{\mu\kappa})_{;\kappa} - \alpha_2'\left(2(\lambda_1 \mathrm{R}^{\mu\kappa})_{;\kappa} - (\lambda_1 \mathrm{R})^{;\kappa}\right) - 6\alpha_3'(\lambda_1 \mathrm{R})^{;\mu}$$

$$= -2(2\alpha_1' + \alpha_2')(\lambda_1 R^{\mu\kappa})_{;\kappa} - (\alpha_2' + 6\alpha_3')(\lambda_1 R)^{;\mu}$$

$$= -2(2\alpha_1' + \alpha_2')\lambda_{1;\kappa} R^{\mu\kappa} - (\alpha_2' + 6\alpha_3')\lambda_1^{;\mu} R - 2(\alpha_1' + \alpha_2' + 3\alpha_3')\lambda_1 R^{;\mu}$$

Due to the very high level of symmetry in cosmology one has, inevitably,
$\lambda_1 = \lambda_1(t)$
$R^{00} = R_0^0(t),\ R^{i0} = 0,\ R = R(t),\ R^{ij} = R_1^1 g^{ij},$
$R_1^1 = (1/3)(R - R_0^0)$
$G^\mu = (G^0(t), 0, 0, 0)$

$G^i[\mathrm{cr}] = 0$ (check!)

$$G^0[\mathrm{cr}] = -2(2\alpha_1' + \alpha_2')\dot{\lambda}_1 R_0^0 - (\alpha_2' + 6\alpha_3')\dot{\lambda}_1 R - 2(\alpha_1' + \alpha_2' + 3\alpha_3')\lambda_1 \dot{R}$$

In our special gauge ($A = 0$) we are dealing, essentially, with Riemannian geometry, and one can rewrite the gravitational Lagrangian in the following way

$$\mathcal{L}_{\mathrm{W}} = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 F_{\mu\nu} F^{\mu\nu}$$

$$= \alpha C^2 + \beta GB + \gamma R^2$$

$C^2 = C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}$, $C_{\mu\nu\lambda\sigma}$ — Weyl tensor = completely traceless part of the curvature tensor $R_{\mu\nu\lambda\sigma}$

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} - \frac{1}{2} R_{\mu\lambda} g_{\nu\sigma} + \frac{1}{2} R_{\mu\sigma} g_{\nu\lambda}$$

$$+\frac{1}{2} R_{\nu\lambda} R_{\mu\sigma} - \frac{1}{2} R_{\nu\sigma} R_{\mu\lambda} + \frac{1}{6} R (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda})$$

$$C^2 = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

GB — Gauss-Bonnet term,

$$GB = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

In 4-dim GB is the full derivative and does not affect the field equations.
It is easy to calculate the coefficients α, β and γ

$$\begin{cases} \alpha + \beta = \alpha_1 \\ -2\alpha - 4\beta = \alpha_2 \\ \frac{1}{3}\alpha + \beta + \gamma = \alpha_3 \end{cases}$$

$2\alpha_1 + \alpha_2 = 2\beta$, $\alpha_1 + 6\alpha_3 = 2(\beta + 3\gamma)$.
We get for G^0:

$$G^i[\mathrm{cr}] = 4\beta\dot{\lambda}_1 R^0_0 - 2(\beta + 3\gamma)\dot{\lambda}_1 R - 6\gamma\lambda_1\dot{R}$$

α does not enter at all, since Weyl tensor is identically zero for any homogeneous and isotropic space-time.

To obtain the left-hand-sides of the gravitational equations one should simply put $\lambda_1 \equiv 1$ and erase "primes" in the expressions for $G^{\mu}[\mathrm{cr}]$ and $T^{\mu\nu}[\mathrm{cr}]$.
Note that in cosmology we need to know only $T^{00} = T_0^0$ and $T = \mathrm{Trace}\, T^{\mu\nu}$, since $T^{0i} = 0$ and $T^{ij} = T_1^1 g^{ij}$, $T_1^1 = (1/3)(T - T_0^0)$. Thus,

$$T = T[\mathrm{part}] + T[\mathrm{cr}]$$

$$T[\mathrm{part}] = \varepsilon - 3p$$

$$T[\mathrm{cr}] = \ddot{\lambda}_1(8\beta' R_0^0 - 4\beta' R - 12\gamma' R)$$

$$-4\dot{\lambda}_1\left(\beta'\frac{\dot{a}}{a}(R + 2R_0^0) + 6\gamma'\dot{R} + 9\gamma'\frac{\dot{a}}{a}R\right) - 12\lambda_1\gamma'(\ddot{R} + 3\frac{\dot{a}}{a}R)$$

$$T_0^0 = T_0^0[\mathrm{part}] + T_0^0[\mathrm{cr}]$$

$$T_0^0[\mathrm{part}] = \varepsilon$$

$$T_0^0[\mathrm{cr}] = 8\gamma'\dot{\lambda}_1\frac{\dot{a}}{a}R_0^0 - 4(\beta' + 3\gamma')\dot{\lambda}_1\frac{\dot{a}}{a}R$$

$$-\gamma'\lambda_1\left(12\frac{\dot{a}}{a}\dot{R} + R(4R_0^0 - R)\right)$$

Gravitational cosmological equations

Vector:

$$-6\gamma\dot{R} = G^0$$

Tensor:

$$-\gamma\left(12\frac{\dot{a}}{a}\dot{R} + R(4R_0^0 - R)\right) = T_0^0$$

$$-12\gamma\left(\ddot{R} + 3\frac{\dot{a}}{a}R\right) = T$$

Self-consistency condition

$$2\frac{(G^0a^3)^{\cdot}}{a^3} = T_0^0 + 3T_1^1 = T$$

It is quite clear that the self-consistency condition is just the consequence of the vector and trace equations. At last, let us write down the expressions for the $\binom{0}{0}$-component of Ricci tensor and the scalar curvature in terms of the scale factor $a(t)$

$$R^0_0 = -3\frac{\ddot{a}}{a}$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right), \quad k = 0, \pm 1$$

How about the equations of motion for the cosmological perfect fluid?

We are left with only one equation plus the law of the particle creation, namely

$$\begin{cases} \dot{\lambda}_1 = -\frac{\varepsilon + p}{n} \\ \frac{(na^3)^{\cdot}}{a^3} = \Phi(\mathrm{inv}) \end{cases}$$

$$\Phi(\mathrm{inv}) = -\frac{4}{3}\beta' R^0_0(2R^0_0 - R) + \gamma' R^2$$

Creation of the universe from nothing

Empty from the very beginning
Vacuum is not absolute, but physical
May or may not it persists?
"Pregnant" vacuum

$$\Phi(\mathrm{inv}) = 0, \quad |\beta'| + |\gamma'| \neq 0$$

$$\frac{4}{3}\beta' R_0^0 (2R_0^0 - R) = \gamma' R^2$$

$$G^0[\mathrm{part}] = T_0^0[\mathrm{part}] = T[\mathrm{part}] = 0$$

$$n = 0$$

$$\dot{\lambda}_1 = -\frac{\varepsilon + p}{n} \quad \Rightarrow$$

1. Non-dust matter
   $\dot{\lambda}_1 = 0 \quad \Rightarrow \quad \lambda_1 = const$
2. Dust matter
   $\dot{\lambda}_1 = -\phi(0) = const \quad \Rightarrow \quad \lambda_1 = -\phi(0)(t - t_0)$

Non-dust pregnancy I

General case: $\beta', \gamma' \neq 0$

$$\lambda_1 = const$$

$$R = \xi R_0^0 \quad \Rightarrow$$

$$(3\gamma'\xi^2 + 4\beta'(\xi - 2))R_0^0 = 0$$

$$-6(\gamma - \gamma'\lambda_1)\dot{R} = 0$$

$$-12(\gamma - \gamma'\lambda_1)\frac{(\dot{R}a^3)^{\cdot}}{a^3} = 0$$

$$-(\gamma - \gamma'\lambda_1)\left\{12\frac{\dot{a}}{a}\dot{R} + R(R - 4R_0^0)\right\} = 0$$

Let, first, $\gamma \neq \gamma'\lambda_1 \quad \Rightarrow \quad \dot{R} = 0$
Either $R = 0 \quad \Rightarrow \quad R_0^0 = 0 \quad \Rightarrow$ Milne universe
Or $\xi = 4 \quad \Rightarrow$ de Sitter for $\beta + 6\gamma' = 0$

Non-dust pregnancy II

General case: $\beta', \gamma' \neq 0$

$$\gamma = \gamma' \lambda_1$$

It is not a special condition, but the solution λ_1

$$\dot{a}^2 + k = C_0 a^{\frac{4}{\xi - 2}}$$

Instability and so on...

Dust pregnancy

$$\lambda_1 = -\phi(0)(t - t_0) \quad \Rightarrow \quad \dot{\lambda}_1 = -\phi < 0, \quad \ddot{\lambda}_1 = 0$$

Surely, there exists the solution with $R = R_0^0 = 0$, i. e., Milne universe.

It can be shown that the only other solution is just the de Sitter universe, with $\xi = 4$ ($R,\ R_0^0 = const$) for $\beta' + 6\gamma' = 0$.

If it is not so, the universe emerging from the vacuum foam, like Aphrodite, immediately starts to produce dust particles !!!

The End