Black hole thermodynamics and S-duality

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Electric-magnetic duality, Dirac string and charge quantization

Maxwell S- duality

$$\vec{E} \to \cos\theta \vec{E} + \sin\theta \vec{B},$$
$$\vec{B} \to -\sin\theta \vec{E} + \cos\theta \vec{B}$$

is violated by Dirac string in Schrodinder eq.

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2\psi + V\psi$$

Transformation of the wave function is required

$$\psi \to e^{-ie\chi}\psi$$
$$\vec{A} \to \vec{A} - \vec{\nabla}\chi$$

Vector-potential for monopole:

$$\vec{A}_N = \frac{g}{4\pi r}\frac{(1-\cos\theta)}{\sin\theta}\hat{e}_\phi \quad 0 \le \theta \le \pi/2$$

North gauge

$$\vec{A}_S = -\frac{g}{4\pi r}\frac{(1+\cos\theta)}{\sin\theta}\hat{e}_\phi \quad \pi/2 \le \theta \le \pi$$

South gauge

related by

$$\vec{A}_N - \vec{A}_S = -\vec{\nabla}\chi, \qquad \chi = -\frac{g}{2\pi}\phi,$$

A_N and A_S on the overlap region is indeed a gauge transformation. We have at $\theta = \pi/2$

$$g = \int_N \vec{B}_N \cdot d\vec{S} + \int_S \vec{B}_S \cdot d\vec{S} = \int_E (\vec{A}_N - \vec{A}_S)\cdot d\vec{l} = \chi(0) - \chi(2\pi)$$

Non- observability by quantum charge leads to Dirac quantization $e^{-ieg} = 1$ or $eg = 2\pi n$: Z2 duality

Non-Abelian generalizations (t'Hooft –Polyakov monopole) ½ $eg = 2\pi n$

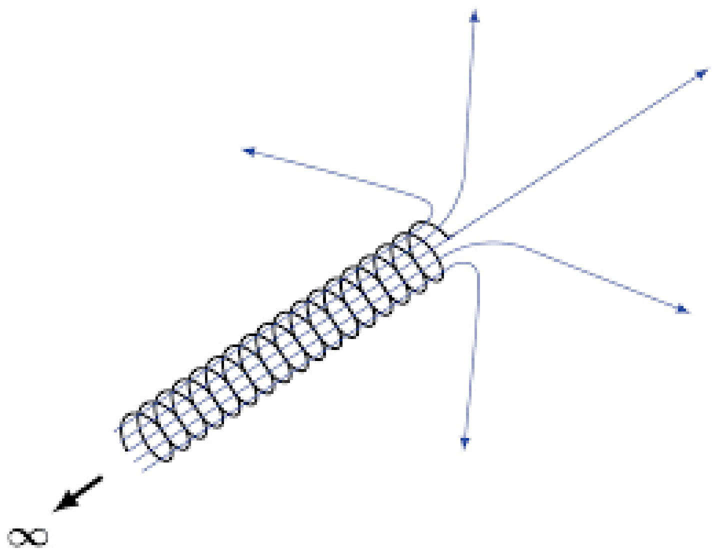
N=2 supersymmetry: central charges $\{Q_{\alpha i}, \overline{Q}_{\beta j}\} = \delta_{ij}\gamma^{\mu}_{\alpha\beta}P_\mu + \delta_{\alpha\beta}U_{ij} + (\gamma_5)_{\alpha\beta}V_{ij}$ $\quad U_{ij} = \epsilon_{ij}vQ_e, \qquad V_{ij} = \epsilon_{ij}vQ_m$

N=4 supersymmetry: dilaton and axion, SL(2,R) classical duality broken to SL(2,Z)

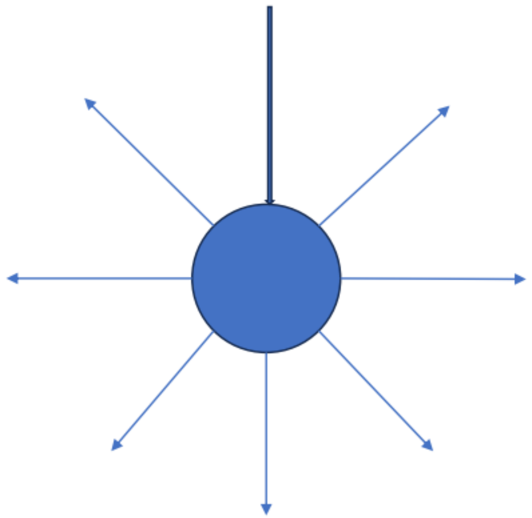
SL(2,R) classical S-duality in IIB supergravity theory relevant to strong-weak AdS/CFT correspondence

Sourceless magnetic fields imitating monopoles

Magnetic monopole from cutting the solenoid carrying magnetic flux



Magnetically charged electrovacuum black hole



Solenoid models “physical” Dirac string

When gravity is turned on, DS becomes heavy

Gravimagnetic field

With special parametrization of linearized metric

$$-ds^2 = -c^2\left(1-\frac{2}{c^2}\Phi\right)dt^2 - \frac{4}{c}(\mathbf{A}\cdot d\mathbf{x})dt + \left(1+\frac{2}{c^2}\Phi\right)\delta_{ij}dx^i dx^j$$

and definitions

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{A}\right), \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A},$$

$$\frac{T^{00}}{c^2} = \rho \qquad\qquad \frac{T^{i0}}{c} = j^i$$

one can present Fierz-Pauli equations in the Maxwell form:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi G\rho, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{B}\right),$$

$$\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\mathbf{B}\right) = 0, \qquad \boldsymbol{\nabla}\times\left(\frac{1}{2}\mathbf{B}\right) = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} + \frac{4\pi}{c}G\,\mathbf{j},$$

The difference is that Maxwell equations can be easily modified to admit magnetic monopoles, the corresponding modification of linearized Einstein equations is more subtle

Gravitational S-duality in linearized theory

Metric Fierz-Pauli theory does not exhibit S-duality but allows for suitable extension

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\,\varepsilon_{\mu\nu\alpha\beta} R^{\alpha\beta}{}_{\rho\sigma}, \qquad R_{\mu\nu\rho\sigma} = -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\tilde{R}^{\alpha\beta}{}_{\rho\sigma},$$

(duality resp. Lorentz indices)

From here find Einstein linearized tensors defining sources

$$\tilde{G}_{\mu\nu} = 8\pi G\Theta_{\mu\nu}, \qquad G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

Gravitational S-duality would mean

$$R_{\mu\nu\rho\sigma} \to \tilde{R}_{\mu\nu\rho\sigma}, \qquad \tilde{R}_{\mu\nu\rho\sigma} \to -R_{\mu\nu\rho\sigma},$$
$$T_{\mu\nu} \to \Theta_{\mu\nu}, \qquad \Theta_{\mu\nu} \to -T_{\mu\nu},$$

To deal with magnetic sources, define

$$\partial_\alpha \Phi^{\alpha\beta}{}_\gamma = -16\pi G\Theta^\beta{}_\gamma, \qquad \Phi^{\alpha\beta}{}_\gamma = -\Phi^{\beta\alpha}{}_\gamma.$$

This generates the cyclic identity for dual R:

$$\bar{\Phi}^{\rho\sigma}{}_\alpha = \Phi^{\rho\sigma}{}_\alpha + \frac{1}{2}(\delta^\rho{}_\alpha\Phi^\sigma - \delta^\sigma{}_\alpha\Phi^\rho), \qquad \Phi^\rho = \Phi^{\rho\alpha}{}_\alpha.$$

$$\begin{aligned}(\tilde{R}_{\mu\nu\alpha\beta} + \tilde{R}_{\mu\beta\nu\alpha} + \tilde{R}_{\mu\alpha\beta\nu}) &= 3\delta^{\rho\sigma\kappa}_{[\nu\alpha\beta]}\tilde{R}_{\mu\rho\sigma\kappa} = -\frac{1}{2}\varepsilon_{\gamma\nu\alpha\beta}(\varepsilon^{\gamma\rho\sigma\kappa}\tilde{R}_{\mu\rho\sigma\kappa}) \\ &= -\frac{1}{2}\varepsilon_{\gamma\nu\alpha\beta}(2R^\gamma{}_\mu - \delta^\gamma{}_\mu R) = 8\pi G\varepsilon_{\nu\alpha\beta\gamma}T^\gamma{}_\mu.\end{aligned}$$

so to maintain duality one has to add the source to initial cyclic identity for consistency:

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = -8\pi G\varepsilon_{\nu\alpha\beta\gamma}\Theta^\gamma{}_\mu,$$
$$\partial_\epsilon\, R_{\gamma\delta\alpha\beta} + \partial_\alpha\, R_{\gamma\delta\beta\epsilon} + \partial_\beta\, R_{\gamma\delta\epsilon\alpha} = 0,$$

$$\tilde{R}_{\mu\nu\alpha\beta} + \tilde{R}_{\mu\beta\nu\alpha} + \tilde{R}_{\mu\alpha\beta\nu} = 8\pi G\varepsilon_{\nu\alpha\beta\gamma}T^\gamma{}_\mu,$$
$$\partial_\epsilon\, \tilde{R}_{\gamma\delta\alpha\beta} + \partial_\alpha\, \tilde{R}_{\gamma\delta\beta\epsilon} + \partial_\beta\, \tilde{R}_{\gamma\delta\epsilon\alpha} = 0,$$

Metric will depend on the choice of magnetic sources

[0909.0542] Why not a di-NUT? or Gravitational duality and rotating solutions (arxiv.org) (Argurio and Dehouck)

Ehlers group: 3D non-linear gravitational S-duality

arXiv:0901.0098v1 [gr-qc] 31 Dec 2008

Consider vacuum Einstein equations $R_{\mu\nu} = 0$. For stationary metrics admitting a time-like Killing vector field, $\mathcal{L}_K g_{\mu\nu} = 0$, $K = \partial_t$, the line element can be presented as

$$ds^2 = -\mathrm{e}^{\xi}(dt + \omega_i dx^i)^2 + \mathrm{e}^{-\xi} h_{ij} dx^i dx^j$$

where the scalar ξ, the three-vector ω_i and the three-dimensional metric h_{ij} depend only on spatial coordinates x^i. Then the equations of motion coincide with those of the three-dimensional gravitating non-linear sigma-model $S_\sigma = \int \left[R_3(h) - \mathcal{G}_{AB}(\Phi)\partial_i\Phi^A\partial_j\Phi^B h^{ij}\right]\sqrt{h}d^3x,$ with two scalar fields $\Phi^A = (\xi, \chi)$, where the twist potential χ is related to the one-form ω_i by the dualization equation $d\chi = -\mathrm{e}^{2\xi} * d\omega$, and the target space is a coset space $SL(2,R)/SO(1,1)$, the corresponding metric being $dl^2 = \mathcal{G}_{AB}d\Phi^A\partial\Phi^B = \frac{1}{2}\left(d\xi^2 + \mathrm{e}^{-2\xi}d\chi^2\right).$ the field equations being invariant under the Ehlers group $SL(2,R)$ acting

$i)$ twist shift (gauge) $\chi \to \chi + \lambda_g$,

$ii)$ scaling $\xi \to \xi + \lambda_s$, $\chi \to \mathrm{e}^{\lambda_s}\chi$,

$iii)$ proper Ehlers transformation $(\chi - i\mathrm{e}^{\xi})^{-1} \to (\chi - i\mathrm{e}^{\xi})^{-1} + \lambda_E$,

the last one generating the Taub-NUT metric from Schwarzschild.

i) and ii) do not change 4d solutions
iii) transforms Schwarzschild to Taub-NUT, it can be interpreted as non-linear S-duality in 3d.
In 4d situation is obscured by the Misner string

Breaking of S-duality by classical solutions: Taub-NUT

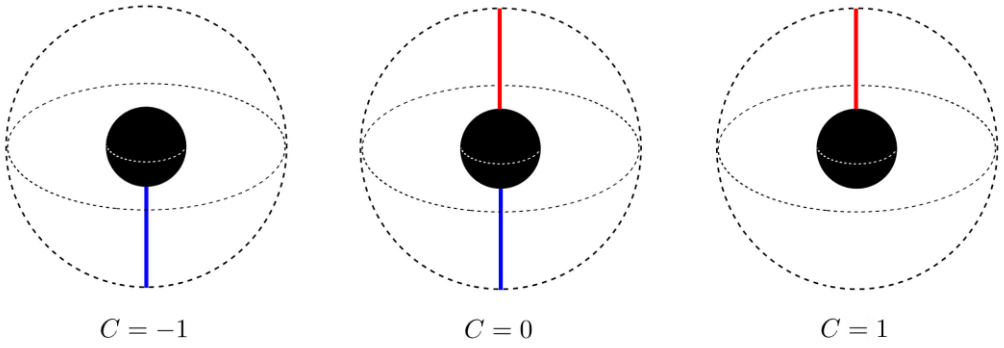
In full non-linear GR this "almost" solution is candidate metric for gravitational dyon with mass M, and S-dual mass "n"

$$ds^2 = -f(r)^2(dt - 2n(\cos\theta + C)\, d\varphi)^2 + \frac{dr^2}{f(r)^2} + (n^2 + r^2)\, d\Omega^2\,, \qquad R_{\mu\nu} = 0$$

except for the polar axis! under linearisa-tion one obtains a distributional string-like source

$$f(r)^2 = \frac{r^2 - 2GMr - n^2}{n^2 + r^2}$$

n is gravimagnetic charge (NUT), **S duality is not manifest in this solutions, though it is the symmetry of the action**



Properties: Non-singular at r=0

C-parameter is Goldstone?

Misner string is shifted by large gauge transformation $t \to t + 2n(C - C')\varphi$.

Misner time identification with period $8\pi n$ is needed to match C=-1 and C=1 charts

(**but thrn Hausdorf axiom violated on the horizon**)

C. W. Misner, "Taub-nut Space As A Counterexample To Almost Anything," (1967)

Bonnor's interpretation: Misner string IS physical singularity (Bonnor, 1969)

G. Clement, D. Gal'tsov and M. Guenouche, "Rehabilitating space-times with NUTs," Phys. Lett. B **750**, 591 (2015)

Supergravities require NUTty solutions for consistency with SUSY algebras (central charges)

NUT wormhole

Clement, DG, Genouch **Phys.Rev. D93 (2016) 024048**

This is Reissner-Nordstrom metric with NUT (know as Brill solution) with four free parameter

"almost"solution of Einstein-Maxwell, except for the axis

$$ds^2 = -f(dt - 2n(\cos\theta + C)\,d\varphi)^2 + f^{-1}dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2)\,,$$

$$A = \Phi(dt - 2n(\cos\theta + C)\,d\varphi)\,;\; f = \frac{(r-m)^2 + b^2}{r^2 + n^2}\,, \quad \Phi = \frac{qr + p(r^2 - n^2)/2n}{r^2 + n^2}\,,$$

$$e^2 = q^2 + p^2 \qquad (b^2 = q^2 + p^2 - m^2 - n^2)\,.$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{4e^4}{(r^2 + n^2)^4}\,;$$

If b^2>0 (overcharged) no horizon, r can be continued to r<0 → wormhole

In the bulk!

No singularity at r=0

Large gauge transformation $t \to t + 2n(C - C')\varphi\,.$

changes distributional sources

In spite of Misner string, isometry algebra is

so(3)x R

$$K_{(x)} = -\frac{2n(1 + C\cos\theta)\cos\varphi}{\sin\theta}\partial_t - \sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi\,,$$

$$K_{(y)} = -\frac{2n(1 + C\cos\theta)\sin\varphi}{\sin\theta}\partial_t + \cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi\,,$$

$$K_{(z)} = \partial_\varphi + 2nC\partial_t\,,$$

$$K_{(t)} = \partial_t\,,$$

Weyl coordinates

Using Weyl coordinates one can explore the structure of the axis

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right]$$

$$U_{,\rho\rho} + \frac{1}{\rho}U_{,\rho} + U_{,zz} = 0$$

This may be recognised as Laplace's equation $\nabla^2 U = 0$ for an axially symmetric function in an unphysical Euclidean 3-space in cylindrical polar coordinates, though the coordinates ρ, z, ϕ here have a different meaning.

Once U is known, one has to integrate a system $\gamma_{,\rho} = \rho\left(U_{,\rho}{}^2 + U_{,z}{}^2\right)$, $\gamma_{,z} = 2\,\rho\, U_{,\rho}\, U_{,z}$

"axis" on which $\rho = 0$ is regular if, and only if, $\gamma \to 0$ as $\rho \to 0$.[2] If this condition is not satisfied for some value or range of z, then some kind of singularity occurs at these points.

Schwarzschild solution in Weyl coordinates reads $U = \frac{1}{2}\log\left(\frac{R_- + z - m}{R_+ + z + m}\right)$

$$e^{2U} = \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m}, \qquad e^{2\gamma} = \frac{(R_+ + R_-)^2 - 4m^2}{4R_+R_-}$$

with $R_\pm^2 = \rho^2 + (z \pm m)^2$

which is formally the Newtonian potential for a finite rod, located along the part of the axis $\rho = 0$ for which $|z| < m$, whose mass per unit length is $\sigma = \frac{1}{2}$. Thus the "rod" has length $2m$ and its total mass is m.

Rod structure of stationary axisymmetric electrovacuum

Clement and DG
1707.01332.pdf (arxiv.org)

Harmark, Reall 2004

In Weyl coordinates x^a, ρ, z, where $x^a = t,\ \varphi$ $\quad ds^2 = G_{ab}(\rho, z)dx^a dx^b + e^{2\nu}(d\rho^2 + dz^2)\,,$

where the Gram matrix G_{ab} and ν are functions of (ρ, z), and $\rho = \sqrt{|\det G|}\,.$

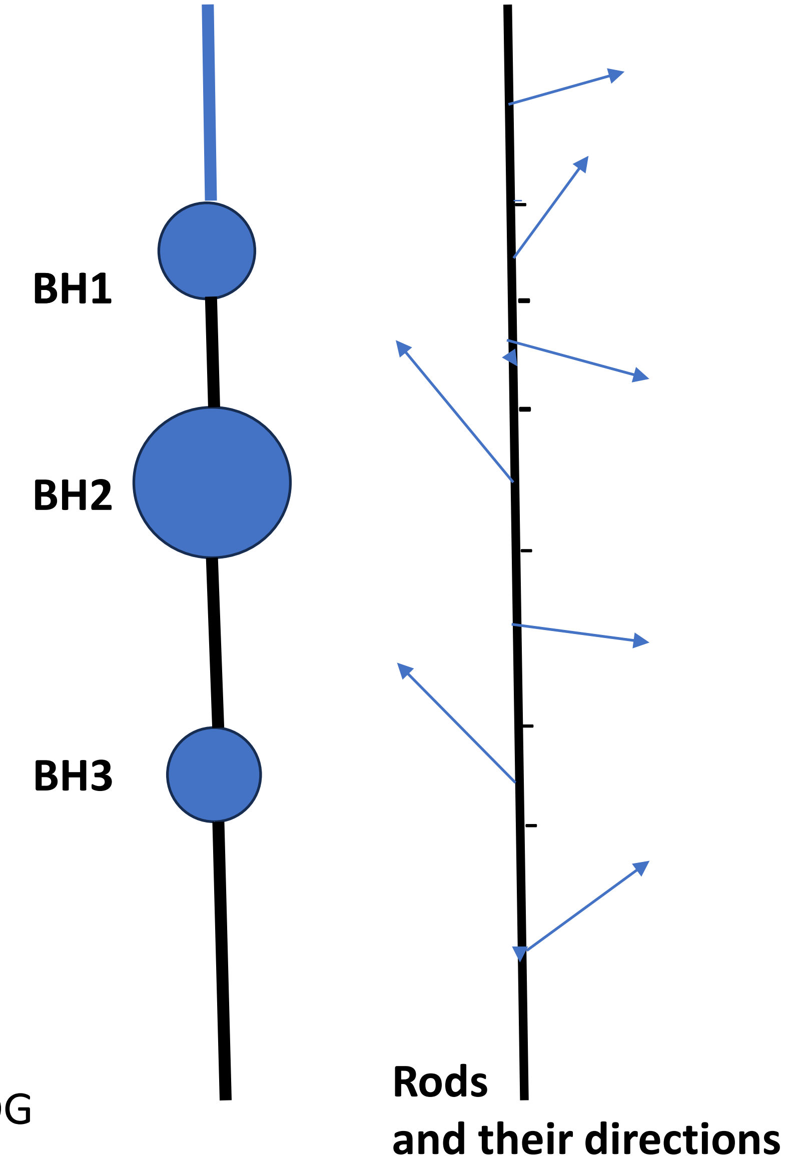
it is clear that the Gram matrix is non-degenerate as long as $\rho > 0$. At $\rho = 0$, it degenerates, so the kernel of the boundary matrix $G(\rho = 0, z)$ becomes nontrivial, i.e., $\dim\ker G(0, z) \geq 1$. It can be proved that, if the kernel has dimension higher than one, there will be a strong curvature singularity on the axis If $\dim\ker G(0, z) = 1$ exactly, except for a finite number of isolated points z_n, $n = 1, \ldots, N$, one encounters only weak distributional singularities on the polar axis, or no singularities at all. The above isolated points, called turning points, will be ordered as $z_1 < z_2 < \cdots < z_N$. The set z_n divides the polar axis z in $N + 1$ intervals $(-\infty, z_1], [z_1, z_2], \ldots, [z_N, +\infty)$ which are called rods (we will label two semi-infinite rods by $n = \pm$, and the remaining finite ones by an index n corresponding to the left bound of the interval). For each rod one defines the eigenvectors $l_n \in \mathbb{R}^2$, belonging to the kernel of $G(0, z)$:

$$l_n = \partial_t + \Omega_n \partial_\varphi \qquad G_{ab}(0, z)l_n^a = 0, \quad z \in [z_n, z_{n+1}]\,.$$

where Ω_n is the constant angular velocity of the rod. In Lorentzian spacetime the norm l_n^2 of this vector can be negative, positive or zero for $\rho > 0$, the associated rod being qualified as timelike, spacelike or null.

The first are event horizons, the second represent cosmic strings or Misner strings

Aligned black holes supported by strings possibly carrying magnetic or gravimagnetic fluxes



Gerard Clement and DG

[2307.06282] The first law for stationary axisymmetric multi-black hole systems (arxiv.org)

Each black hole has M,N,Q,P, a as parameters and the corresponding rod direction is timelike

Each string corresponds to spacelike rod

Each rod direction specifies certain spacetime Killing vector (linear combination of k and m) and the axis serves Killing horizons for them. Parameters of rods are constant surface gravity and angular velocity

External rods correspond to Misner strings if the sum of NUT parameters of all BH is non-zero

Using Komar integrals one can calculate parameters M, Q, a of each black hole, but not P and N. These are computed using Komar integrals for string segments. When there are magnetic and gravimagnetic fluxes, parametes P and N can not be prescribed to black holes, they are delocalized. Stings also can carry electric charges

Conserved charges of rods

The spacetime Killing vectors associated with the rods share with them the same causal nature (timelike for horizons and spacelike for defects) and they become null on the rods themselves. Thus both timelike and spacelike rods are Killing horizons with certain surface gravities, which together with the angular velocities Ω_n are constant along the rods.

The total Komar mass, angular momentum and electric charge of a stationary axisymmetric configuration are given by the integrals over Σ_∞:

$$M = \frac{1}{4\pi} \oint_{\Sigma_\infty} D^\nu k^\mu d\Sigma_{\mu\nu}, \quad J = -\frac{1}{8\pi} \oint_{\Sigma_\infty} D^\nu m^\mu d\Sigma_{\mu\nu}, \quad Q = \frac{1}{4\pi} \oint_{\Sigma_\infty} F^{\mu\nu} d\Sigma_{\mu\nu},$$

where $k^\mu = \delta^\mu_t$ and $m^\mu = \delta^\mu_\varphi$ are the Killing vectors associated with time translations and rotations around the z-axis, D^ν is the covariant derivative and $F^{\mu\nu}$ is the Maxwell tensor. Using the Einstein-Maxwell equations and the Gauss-Ostrogradsky theorem, the Komar charges can be expressed as the sums of integrals over the various rods Σ_n (two-surfaces spanned by the coordinates z, φ):

$$M = \sum_n M_n, \quad J = \sum_n J_n, \quad Q = \sum_n Q_n$$

$$\begin{aligned} M_n &= \frac{1}{8\pi} \oint_{\Sigma_n} \left[g^{ij} g^{ta} \partial_j g_{ta} + 2(A_t F^{it} - A_\varphi F^{i\varphi}) \right] d\Sigma_i, \\ J_n &= -\frac{1}{16\pi} \oint_{\Sigma_n} \left[g^{ij} g^{ta} \partial_j g_{\varphi a} + 4 A_\varphi F^{it} \right] d\Sigma_i, \\ Q_n &= \frac{1}{4\pi} \oint_{\Sigma_n} F^{ti} d\Sigma_i. \end{aligned}$$

the axisymmetric Weyl-Papapetrou parametrization

$$\begin{aligned} ds^2 &= -F(dt-\omega d\varphi)^2 + F^{-1}[e^{2k}(d\rho^2+dz^2)+\rho^2 d\varphi^2], \\ A &= v dt + A_\varphi d\varphi, \end{aligned}$$

the imaginary parts of the complex electromagnetic and gravitational Ernst potentials, $u = \operatorname{Im}\psi$ (the scalar magnetic potential) and $\chi = \operatorname{Im}\mathcal{E}$ (the twist potential) are defined up to an additive constant by the dualizations

$$\begin{aligned} \partial_i u &= F\rho^{-1}\epsilon_{ij}\left(\partial_j A_\varphi + \omega\partial_j v\right), \\ \partial_i \chi &= -F^2\rho^{-1}\epsilon_{ij}\partial_j\omega + 2(u\partial_i v - v\partial_i u), \end{aligned}$$

Using these to evaluate the integrands

$$\begin{aligned} M_n &= \frac{1}{4}\omega_n\delta_n\chi + \frac{1}{2}\delta_n(A_\varphi\, u), \\ J_n &= \frac{1}{4}\omega_n\left\{-L_n + \omega_n\left(\delta_n\chi/2 - \Phi_n\delta_n u\right) + \delta_n(A_\varphi\, u)\right\}, \\ Q_n &= \frac{1}{2}\omega_n\delta_n u. \end{aligned}$$

where for any function of two variables $f(\rho,z)$ the quantity $\delta_n f \equiv f(0,z_{n+1}) - f(0,z_n)$ i variation between two ends z_{n+1} and z_n of the rod n, while $L_n \equiv z_{n+1} - z_n$ is rod's length related to the rescaled surface area of the Killing horizon $L_i = 4\kappa_i\mathcal{A}_i$.

$$\mathcal{A}_n = \frac{1}{8\pi}\oint d\varphi\int_{z_n}^{z_{n+1}}\sqrt{|g_{zz}g_{\varphi\varphi}|}dz = \frac{1}{4}\int_{z_n}^{z_{n+1}}|e^k\omega|dz \qquad \kappa_i = |e^{-k_i}\Omega_i|$$

is the constant Killing horizon surface gravity.

Smarr mass formulas and the first law of black hole mechanics

Using this, we get $$M_n = 2\Omega_n J_n + 2\kappa_n \mathcal{A}_n + \Phi_n Q_n$$

valid for each Killing horizon, black hole event horizon or Misner string. Finally, adding together these individual Smarr relations and splitting summation indices n in two sets h, s enumerating black hole and string contributions respectively, we obtain the global Smarr relation for the system:

$$M = \sum_{h=1}^{N} \left(2\Omega_h J_h + 2\kappa_h \mathcal{A}_h + \Phi_h Q_h\right) + \sum_{s=1}^{N-1} \left(2\Omega_s J_s + 2\kappa_s \mathcal{A}_s + \Phi_s Q_s\right),$$

where the first sum relates to constituent black holes, and the second to defects.

The first law for such a system is **(NUTs are included)**

$$dM = \sum_{h=1}^{N} \left(\Omega_h dJ_h + \kappa_h d\mathcal{A}_h + \Phi_h dQ_h\right) + \sum_{s=1}^{N-1} \left(\Omega_s dJ_s + \kappa_s d\mathcal{A}_s + \Phi_s dQ_s\right),$$

where M is the total mass (sum of the horizon and string masses). It expresses the differential of the total mass (energy) in terms of differentials of extensive (additive) quantities J_i, $\mathcal{A}_i$, Q_i.

Misner strings (among other defects) enter the second sum. Their contribution is purely mechanical, though is put in the form analogous to entropic terms in the first sum. No S-duality for magnetic and gravimagnetic charges

G.Clement and DG, Phys lett B 2023

The first law for NUTless dyonic multi-black holes

$$dM = \sum_{h=1}^{N} (\Omega_h dJ_h + \kappa_h d\mathcal{A}_h + \Phi_h dQ_h) + \sum_{s=1}^{N-1} (\Psi_s dP_s - \lambda_s d\mu_s)$$

Contrary to naive expectations, the first law (4.8) for axisymmetric arrays of NUTless dyonic black holes is not generically invariant under electric-magnetic duality. It is true that the magnetic flux through the Dirac string S_n connecting the two black holes H_n and H_{n+1} can be thought of as the difference $P_{S_n} = P_{H_{n+1}} - P_{H_n}$ so that

$$dM = \sum_{h=1}^{N} \left(\Omega_h dJ_h + \kappa_h d\mathcal{A}_h + \Phi_h dQ_h + \tilde{\Phi}_h dP_h\right) - \sum_{s=1}^{N-1} \lambda_s d\mu_s$$

where $\tilde{\Phi}_h = 2u_h - u_{h-1} - u_{h+1} + \lambda$

the "potentials" $\tilde{\Phi}_h$ thus formally defined do not characterize the black hole horizon H_n, as they depend in a non-local fashion on the values of the magnetic potential u on several horizons.

Rotating Kerr-Newman wormhole with NUT

Consider Kerr-Newman-NUT solution

2210.08913.pdf (arxiv.org)

$$
\begin{aligned}
ds^2 &= -\frac{\Delta}{\Sigma}(dt - \alpha d\varphi)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\Sigma}(\beta d\varphi - a dt)^2 \\
&= -\frac{f}{\Sigma}(dt - \omega d\varphi)^2 + \Sigma\left[\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{f}\sin^2\theta d\varphi^2\right],
\end{aligned}
$$

$$
\begin{aligned}
&\alpha = a\sin^2\theta - 2n\cos\theta, \quad \beta = r^2 + n^2 + a^2, \\
&\Delta = r^2 - 2mr + e^2 - n^2 + a^2, \\
&\Sigma = \beta - a\alpha = r^2 + (n + a\cos\theta)^2, \\
&f = \Delta - a^2\sin^2\theta, \quad \omega f = \alpha\Delta - a\beta\sin^2\theta,
\end{aligned}
$$

This is rotating "almost" solution of EM equations with five free parameters: (q, p, m, n, a). Different sectors in parameter space correspond to BH, naked singularities or wormholes. It has Dirac and Misner string singularities

where $e^2 = q^2 + p^2$, with q and p the electric and magnetic charges, and m, n and a are the mass, NUT and rotational parameters.

The outer root of the equation $\Delta = 0$, $r_h = m + \sqrt{m^2 + n^2 - a^2 - e^2}$ defines the location of the event horizon, which exists and is non-degenerate if $m^2 + n^2 - a^2 - e^2 > 0$, exists and is degenerate (extremal) if $m^2 + n^2 - a^2 - e^2 = 0$, and does not exist for $a^2 + e^2 - m^2 - n^2 \equiv b^2 > 0$

In this overcharged case it is:

naked ring singularity at $r = 0, a\cos\theta = n$, if $|n| < a$ or is non-singular if $|n| > a$ in which case the radial variable r extends to the whole real axis, leading to a wormhole topology with two spacelike asymptotics $r = +\infty$ and $r = -\infty$.

For the KNN wormhole the metric function

$$\Delta(r) = (r - m)^2 + b^2$$

is always positive, so there is no horizon. The point $r = m$ corresponds to its minimal value $\Delta_0 = b^2$. The metric function $f(r, \theta)$ is also positive definite for $b > a$. However, for $a > b$,
the Killing vector ∂_t becomes null on the boundary of the ergoregion $r < r_e$ where

$$r_e = m + \sqrt{a^2 \sin^2 \theta - b^2}.$$

This boundary ends on non-zero value of the polar angle

There is an ergosphere
but no superradiance!

$$\pi - \theta_e > \theta > \theta_e, \qquad \theta_e = \arcsin\left(\frac{b}{a}\right).$$

Thus the ergoregion $r < r_e$ is bounded by the cones $\theta = \theta_e$, $\theta = \pi - \theta_e$. Its maximal radius at $\theta = \pi/2$ is

$$r_e = m + \sqrt{a^2 \sin^2 \theta - b^2}.$$

Misner String violates Null Energy Condition

wormhole configurations requires violation of the null energy condition

$$R_{\mu\nu}k^\mu k^\nu \geq 0 \quad \text{for some null vector field } k(x)\ (g_{\mu\nu}k^\mu k^\nu = 0)$$

How to extract distributional hidden sources?

Stationary metric in Kaluza-Klein form $ds^2 = -F\left(dt - \omega_i dx^i\right)^2 + F^{-1}h_{ij}dx^i dx^j$

$$R^i_t = \frac{F}{2\sqrt{h}}\epsilon^{ijk}\tau_{k,j}, \qquad \tau^i = \epsilon^{ijk}\frac{F^2}{\sqrt{h}}\partial_j\omega_k.$$

Near a Misner string $u \equiv \sin\theta = 0$, we have

$$h \simeq dr^2 + \Delta(du^2 + u^2 d\varphi^2), \quad \omega \simeq \mp 2n d\varphi.$$

Transforming to local coordinates $x^1 = r$, $x^2 = u\cos\varphi$, $x^3 = u\sin\varphi$

$$h \simeq (dx^1)^2 + \Delta\left[(dx^2)^2 + (dx^3)^2\right], \quad \omega \simeq \pm 2n\epsilon_{1ij}\partial_j \ln u\, dx^i$$

$(u^2 = (x^2)^2 + (x^3)^2)$. Accordingly,

$$\tau_1 = \tau^1 \simeq \mp\frac{2n\Delta}{\Sigma^2}\nabla^2 \ln u = \mp\frac{4\pi n\Delta}{\Sigma^2}\delta^2(x)$$

$\Rightarrow$

$$\sqrt{|g|}(R_S)^i_t = \mp\frac{2\pi n\Delta}{\Sigma^2}\epsilon^{1ij}\partial_j\delta^2(x)$$

$$X_S(r) \equiv \int\sqrt{|g|}(R_S)^i_t k^t k_i d^3x = -8\pi n^2\int_{-\infty}^{+\infty}\frac{{\lambda_S}^2\Delta^2\beta}{{\Sigma_S}^2}dr$$

where $k^t = \lambda\beta\sin\theta,\ k^r = \varepsilon\lambda\Delta\sin\theta,$
$k^\theta = 0,\ k^\varphi = \lambda a\sin\theta$

Conclusions

Electrovacuum black holes (RN, KN) are “charges without charges”

Consequently Dirac and Misner strings are implied by flux conservation

Though electrically and magnetically charged black holes are S-symmetric
the corresponding first law are different because of Dirac strings

The same is true for gravimagnetic black holes (with NUTs)

Gravimagnetic charge can turn overcharged naked singularities into wormholes

Exotic matter sources are involved, violating NEC in the case of NUT wormholes

First law exhibits violation of S-duality in multicenter solutions