Electron parton distribution functions

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Outline

Motivation

- The Standard Model now is the most succesful theory of elementary particles and their interactions
- But! Many phenomena are not explained by the SM (neutrino masses, baryon asymmetry, dark matter, dark energy...)
- Experimental data agree with the SM predictons well
- We need experiments with higher precision, sensitivity or energy to see someting beyond the SM
- To predict and describe results of experiments we need very accurate theoretical predictions $\rightarrow$ higher order corrections

Parton distribution functions approach

- Based on perturbation theory and R. Feynmann parton theory
- Came from QCD to QED
- Allows to calculate the most significant (logarithmic) corrections
- Expansion in powers of coupling constant and the large logarithm

Large logarithm

$$L = \ln \frac{\mu^2}{\mu_0^2}$$

μ - factorization scale, μ_0 - renormalization scale (in QED $\mu_0 = m_e$)

- Leading logarithmic approximation (LL or leading order - LO) - $\alpha^n L^n$
- Next-to-leading logarithmic approximation (NLL or next-to-leading order - NLO) - $\alpha^n L^{n-1}$

QED corrections

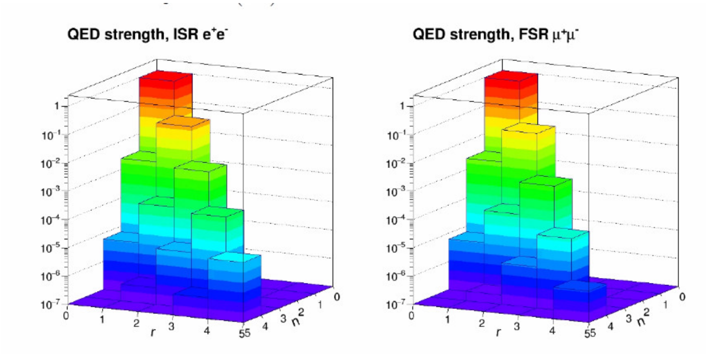


Fig. from Jadach, Skrzypek, arXiv: 1903.09895
Corrections of the order $\alpha^r L^n$

Parton distribution functions approach

Parton distribution functions (PDFs)

A function $D_{ia}(x, s)$ describes the density of the distribution of the massles parton of type i in the initial massive parton a. x is the energy fraction of the parton relative to the total energy of the particle which emitted it. Structure functions correspond to transition of a massive particle to a massless, and fragmentation functions to transition from massless particle to a massive one, which can be observed.

Splitting functions

$P_{ij}(x)$ describes the probability density of a transition of a parton j to a parton i with the energy fraction x.

Splitting functions and PDFs are independent of the process

PDF evolution equation in QED

Convolution operation

$$(f \otimes g)(x) \equiv \int_0^1 dz \int_0^1 dy\ f(z)g(y)\delta(x - yz) = \int_x^1 \frac{dz}{z} f(z) g(\frac{x}{z})$$

$$f(x) = \lim_{\Delta \to 0} \Big(f_\Theta(x)\Theta(1 - x - \Delta) + f_\Delta \delta(1 - x) \Big)$$

$$\int_z^1 dx [f(x)]_+ g(x) = \int_0^1 dx f(x) \Big[g(x)\Theta(x - z) - g(1) \Big]$$

$$f_\Delta = - \int_0^{1-\Delta} f_\Theta(z) dz$$

$$\left(f \otimes g \right)_\Theta (z) = \lim_{\Delta \to 0} \Big\{ \int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x} f_\Theta(x) g_\Theta \left(\frac{z}{x} \right) + f_\Delta g_\Theta(z) + f_\Theta(z) g_\Delta \Big\}$$

PDF evolution equation in QED

$$D_{ba}(x,\mu^2,\mu_0^2) = \delta(1-x)\delta_{ba} + \sum_{i=e,\bar{e},\gamma} \int\limits_{\mu_0^2}^{\mu^2} \frac{dt\alpha(t)}{2\pi t} \int\limits_{x}^{1} \frac{dy}{y} D_{ia}(y,t,\mu_0) P_{bi}\left(\frac{x}{y}\right)$$

Analog of DGLAP (Dokshitzer-Gribov-Lipatov-ALtarelli-Parisi) equations on QCD (*G. Altarelli, G. Parisi,Nuclear Physics B. 126 (2); Yu. L. Dokshitzer. Sov. Phys. JETP 46:641 (1977); V. N. Gribov, L. N. Lipatov. Sov. J. Nucl. Phys. 15:438 (1972)*)
Derived for QED by E.A. Kuraev and V.S. Fadin (*E.A. Kuraev, V.S. Fadin, 1985, Sov. J. Nucl. Phys. 41*)

Iterative solution

Equations are solved using iterative method:

$$D_{ee}^{(k)} = D_{ee}^{(0)} + \frac{\alpha}{2\pi}\left(P_{ee} \otimes D_{ee}^{(k-1)} + P_{e\gamma} \otimes D_{\gamma e}^{(k-1)} + P_{e\bar{e}} \otimes D_{\bar{e}e}^{(k-1)}\right)$$

$$D_{e\gamma}^{(k)} = D_{e\gamma}^{(0)} + \frac{\alpha}{2\pi}\left(P_{ee} \otimes D_{e\gamma}^{(k-1)} + P_{e\bar{e}} \otimes D_{\bar{e}\gamma}^{(k-1)} + P_{e\gamma} \otimes D_{\gamma\gamma}^{(k-1)}\right)$$

$$P_{ji}(x) = P_{ji}^{(0)}(x) + \frac{\alpha}{2\pi} P_{ji}^{(1)}(x) + \mathcal{O}(\alpha^2)$$

Initial conditions:

$$D_{ee}^{(0)}(x,\mu^2) = \delta(1-x) + \frac{\alpha}{2\pi} d_{ee}^{(1)}, \quad D_{\bar{e}\gamma}^{(0)}(x,\mu^2) = 0, \quad D_{\gamma\gamma}^{(0)}(x,\mu^2) = \delta(1-x)\frac{\beta}{3}$$

$$D_{e\gamma}^{(0)}(x,\mu^2) = \frac{\alpha}{2\pi} d_{e\gamma}^{(1)}(x), \quad D_{e\bar{e}}^{(0)}(x,\mu^2) = 0$$

Electron PDFs

$$D_{ee}^{(\mathrm{I})}(x,\mu) = \delta(1-x) + \frac{\alpha}{2\pi} d_{ee}^{(1)}(x) + \frac{\alpha}{2\pi} L P_{ee}^{(0)}$$

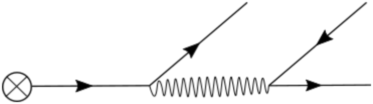
$$D_{ee}^{(\mathrm{II})}(x,\mu) = D_{ee}^{(\mathrm{I})}(x,\mu) + \left(\frac{\alpha}{2\pi}\right)^2 L \left(d_{\gamma e}^{(1)}(x) \otimes P_{e\gamma}^{(0)} + P_{ee}^{(1)} - \frac{10}{9} P_{ee}^{(0)} + P_{ee}^{(0)} \otimes d_{ee}^{(1)}(x) \right)$$
$$+ \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{2} P_{\bar{e}e}^{(0)} \otimes P_{e\bar{e}}^{(0)} + \frac{1}{2} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{3} P_{ee}^{(0)} + \frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \right)$$

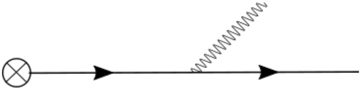
$$D_{ee}^{(\mathrm{III})}(x,\mu) = D_{ee}^{(\mathrm{II})}(x,\mu) + \left(\frac{\alpha}{2\pi}\right)^3 L^2 \left(\frac{1}{2} P_{\gamma e}^{(1)} \otimes P_{e\gamma}^{(0)} + \frac{1}{2} P_{\bar{e}e}^{(0)} \otimes P_{e\bar{e}}^{(1)} + \frac{1}{3} d_{\gamma e}^{(1)} \otimes P_{e\gamma}^{(0)} \right.$$
$$+ \frac{1}{2} d_{\gamma e}^{(1)} \otimes P_{\gamma\gamma}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{2} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(1)} - \frac{10}{9} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{2}{3} P_{ee}^{(1)} + \frac{1}{2} P_{\bar{e}e}^{(1)} \otimes P_{e\bar{e}}^{(0)}$$
$$- \frac{10}{9} P_{\bar{e}e}^{(0)} \otimes P_{e\bar{e}}^{(0)} + \frac{1}{2} d_{\gamma e}^{(1)} \otimes P_{\bar{e}\gamma}^{(0)} \otimes P_{e\bar{e}}^{(0)} + \frac{1}{2} d_{ee}^{(1)} \otimes P_{\bar{e}e}^{(0)} \otimes P_{e\bar{e}}^{(0)} + \frac{1}{2} d_{ee}^{(1)} \otimes P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}$$
$$- \frac{13}{54} P_{ee}^{(0)} + \frac{1}{2} P_{ee}^{(0)} \otimes d_{\gamma e}^{(1)} \otimes P_{e\gamma}^{(0)} + P_{ee}^{(0)} \otimes P_{ee}^{(1)} + \frac{1}{3} P_{ee}^{(0)} \otimes d_{ee}^{(1)} - \frac{10}{9} P_{ee}^{(0)} \otimes P_{ee}^{(0)}$$
$$\left. + \frac{1}{2} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \otimes d_{ee}^{(1)} \right) + \left(\frac{\alpha}{2\pi}\right)^3 L^3 \left(\frac{1}{3} P_{\bar{e}e}^{(0)} \otimes P_{e\bar{e}}^{(0)} + \frac{1}{6} P_{\bar{e}\bar{e}}^{(0)} \otimes P_{e\bar{e}}^{(0)} \otimes P_{e\bar{e}}^{(0)} + \frac{4}{27} P_{ee}^{(0)} \right.$$
$$+ \frac{1}{6} P_{\gamma e}^{(0)} \otimes P_{e\bar{e}}^{(0)} \otimes P_{\bar{e}\gamma}^{(0)} + \frac{1}{3} P_{ee}^{(0)} \otimes P_{\bar{e}e}^{(0)} \otimes P_{e\bar{e}}^{(0)} + \frac{1}{6} P_{\gamma\bar{e}}^{(0)} \otimes P_{\bar{e}e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{3} P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)}$$
$$\left. + \frac{1}{6} P_{\gamma e}^{(0)} \otimes P_{\gamma\gamma}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{3} P_{ee}^{(0)} \otimes P_{\gamma e}^{(0)} \otimes P_{e\gamma}^{(0)} + \frac{1}{3} P_{ee}^{(0)} \otimes P_{ee}^{(0)} + \frac{1}{6} P_{ee}^{(0)} \otimes P_{ee}^{(0)} \otimes P_{ee}^{(0)} \right)$$

Electron PDFs

D_{ee} consists of 4 parts:

- Pure photonic
- Singlet
- Non-singlet
- Interference





Electron PDFs

$$
\begin{aligned}
\mathcal{D}_e^{(\gamma)}(x) &= \delta(1-x) + \frac{\alpha}{2\pi} d_1(x,\mu_0,m_e) + \frac{\alpha}{2\pi} L_f P_{ee}^{(0)}(x) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 \left(\frac{1}{2} L_f^2 P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x) + L_f P_{ee}^{(0)} \otimes d_1(x,\mu_0,m_e) + L_f P_{ee}^{(1,\gamma)}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 \frac{1}{6} L_f^3 P_{ee}^{(0)} \otimes P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x), \\
\mathcal{D}_e^{(\mathrm{NS})}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 \left(\frac{1}{3} L_f^2 P_{ee}^{(0)}(x) + L_f P_{ee}^{(1,\mathrm{NS})}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L_f^3 \left(\frac{1}{3} P_{ee}^{(0)} \otimes P_{ee}^{(0)}(x) + \frac{4}{27} P_{ee}^{(0)}(x)\right), \\
\mathcal{D}_e^{(\mathrm{S})}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 \left(\frac{1}{2} L_f^2 P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}(x) + L_f P_{ee}^{(1,\mathrm{S})}(x)\right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L_f^3 \left(\frac{1}{3} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)} \otimes P_{ee}^{(0)}(x) - \frac{1}{9} P_{e\gamma}^{(0)} \otimes P_{\gamma e}^{(0)}(x)\right), \\
\mathcal{D}_e^{(\mathrm{int})}(x) &= \left(\frac{\alpha}{2\pi}\right)^2 L_f P_{ee}^{(1,\mathrm{int})}(x), \qquad L_f \equiv \ln \frac{\mu_f^2}{m_e^2},
\end{aligned}
$$

Electron PDFs

$$D^{\mathrm{I}}_{e\gamma} = \frac{\alpha}{2\pi} d^{(1)}_{e\gamma} + \frac{\alpha}{2\pi} L P^{(0)}_{e\gamma}$$

$$D^{\mathrm{II}}_{e\gamma} = D^{\mathrm{I}}_{e\gamma} + \left(\frac{\alpha}{2\pi}\right)^2 L\Big(P^{(1)T}_{e\gamma} - \frac{10}{9} P^{(0)}_{e\gamma} + P^{(0)}_{e\bar{e}} \otimes d^{(1)}_{e\gamma} +$$

$$+ P^{(0)}_{ee} \otimes d^{(1)}_{e\gamma}\Big) + \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{3} P^{(0)}_{e\gamma} + \frac{1}{2} P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{e\gamma} + \frac{1}{2} P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma}\right)$$

$$D^{\mathrm{III}}_{e\gamma} = D^{\mathrm{II}}_{e\gamma} + \left(\frac{\alpha}{2\pi}\right)^3 L^2 \Big(\frac{1}{2} \otimes P^{(1)T}_{e\bar{e}} \otimes P^{(0)}_{\bar{e}\gamma} + \frac{1}{3} P^{(1)T}_{e\gamma} - \frac{13}{54} P^{(0)}_{e\gamma}$$

$$+ \frac{1}{2} P^{(1)T}_{\gamma\gamma} \otimes P^{(0)}_{e\gamma} + \frac{1}{2} P^{(0)}_{\gamma\gamma} \otimes P^{(1)T}_{e\gamma} - \frac{10}{9} P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{e\gamma} + \frac{1}{2} d^{(1)}_{e\gamma} \otimes P^{(0)}_{\gamma\bar{e}} \otimes P^{(0)}_{e\gamma}$$

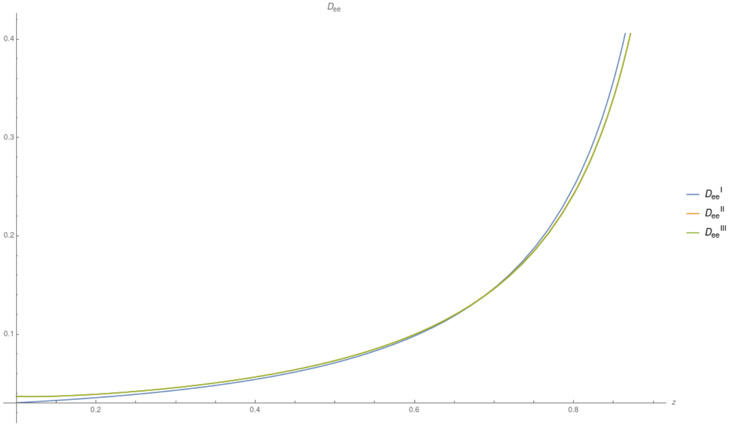
$$+ \frac{1}{2} P^{(0)}_{\gamma e} \otimes d^{(1)}_{e\gamma} \otimes P^{(0)}_{e\gamma} + \frac{1}{2} P^{(1)T}_{ee} \otimes P^{(0)}_{e\gamma} + \frac{1}{2} P^{(0)}_{ee} \otimes P^{(1)T}_{e\gamma} - \frac{10}{9} P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma}$$

$$+ \frac{1}{3} P^{(0)}_{ee} \otimes d^{(1)}_{e\gamma} + \frac{1}{2} P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes d^{(}_{e\gamma}1)\Big) + \left(\frac{\alpha}{2\pi}\right)^3 L^3 \Big(\frac{4}{27} P^{(0)}_{e\gamma}$$

$$+ \frac{1}{3} P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{e\gamma} + \frac{1}{6} P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{e\gamma} + + \frac{1}{6} P^{(0)}_{\gamma\bar{e}} \otimes P^{(0)}_{e\gamma} \otimes P^{(0)}_{\bar{e}\gamma}$$

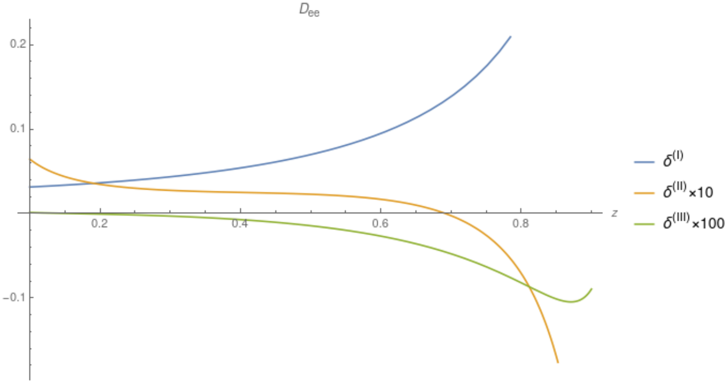
$$+ \frac{1}{6} P^{(0)}_{\gamma e} \otimes P^{(0)}_{e\gamma} P^{(0)}_{e\gamma} + \frac{1}{3} P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma} + \frac{1}{6} P^{(0)}_{ee} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{e\gamma} + \frac{1}{6} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma}\Big)$$

D_{ee}

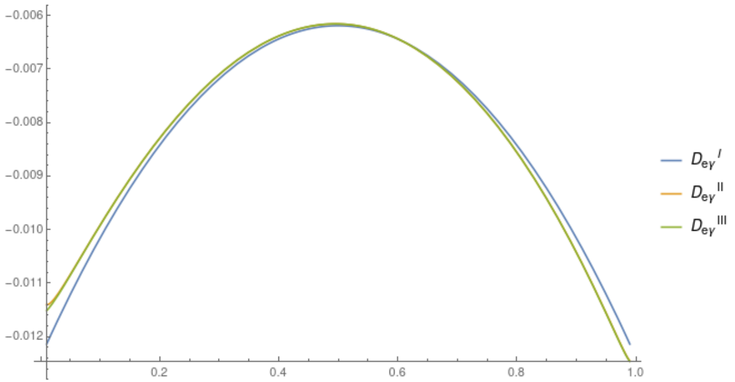


D_{ee}

$$\delta^{(I)} = D^{(\mathrm{I})} - D^0$$
$$\delta^{(II)} = D^{(\mathrm{II})} - D^{(\mathrm{I})}$$
$$\delta^{(III)} = D^{(\mathrm{III})} - D^{(\mathrm{II})}$$

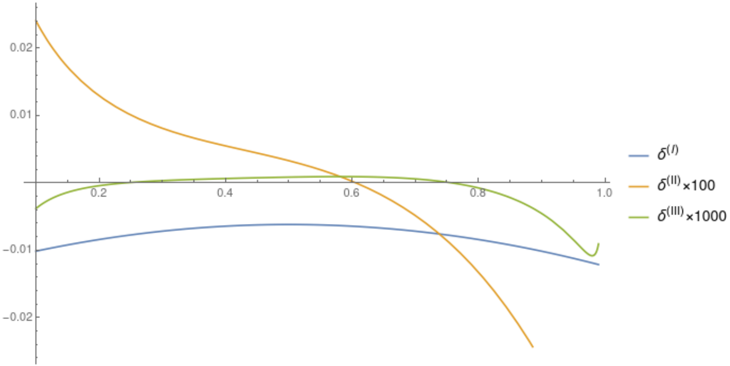


$D_{e\gamma}$



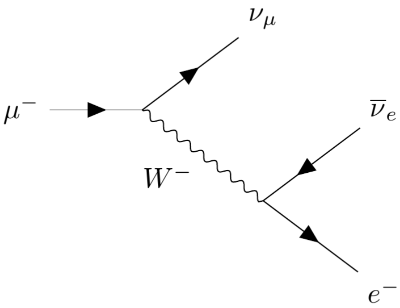
−0.006
−0.007
−0.008
−0.009
−0.010
−0.011
−0.012
0.2
0.4
0.6
0.8
1.0
Deγ I
Deγ II
Deγ III

$D_{e\gamma}$



Applications

Muon decay



Experiments: $(g-2)_\mu$, Mu2e, Mu3e, COMET
Potential new physics: Dirac or Majorana nature of ν, rare and forbidden decays...

Corrections to muon decay spectrum to the order α^3

$$F(x) + cP_\mu G(x),$$

P_μ - polarization vector, c - $\cos\theta$

$$F(x) = \left(f_e^{(0)}(z) + \frac{\alpha}{2\pi} f_e^{(1)}(z)\right) \otimes \left[D_{ee}^{(\mathrm{III})}\right]_T + \left(f_\gamma^{(0)}(z) + \frac{\alpha}{2\pi} f_\gamma^{(1)}(z)\right) \otimes \left[D_{e\gamma}^{(\mathrm{II})}\right]_T$$

Energy spectrum functions (*T. Kinoshita, A. Sirlin. Phys.Rev.Lett. 2 (1959) 4, 177*):

$$f_e^{(0)}(z) = z^2(3-2z), \quad f_\gamma^{(0)}(z) = 0$$

$$f_e^{(1)}(z) = 2z^2(2z-3)\big(4\zeta(2) - 4\mathrm{Li}_2(z) + 2\ln z^2 - 3\ln z \ln(1-z)$$

$$-\ln(1-z)^2\big) + \left(\frac{5}{3} - 2z - 13z^2 + \frac{34}{3}z^3\right)\ln(1-z)$$

$$+\left(\frac{5}{3} + 4z - 2z^2 - 6z^3\right)\ln z + \frac{5}{6} - \frac{23}{3}z - \frac{3}{2}z^2 + \frac{7}{3}z^3$$

$$f_\gamma^{(1)}(z) = \ln z\left(-\frac{10}{3} + \frac{2}{z} + 4z\right) + \ln(1-z)\left(-\frac{5}{3} + \frac{1}{z} + 2z - 2z^2 + \frac{2}{3}z^3\right)$$

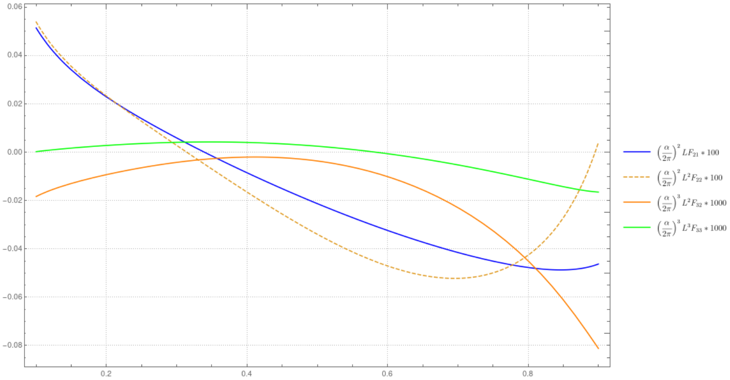
$$+\frac{1}{3} - \frac{1}{z} + \frac{35}{12}z - 2z^2 - \frac{1}{4}z^3$$

Corrections to muon decay spectrum to the order α^3

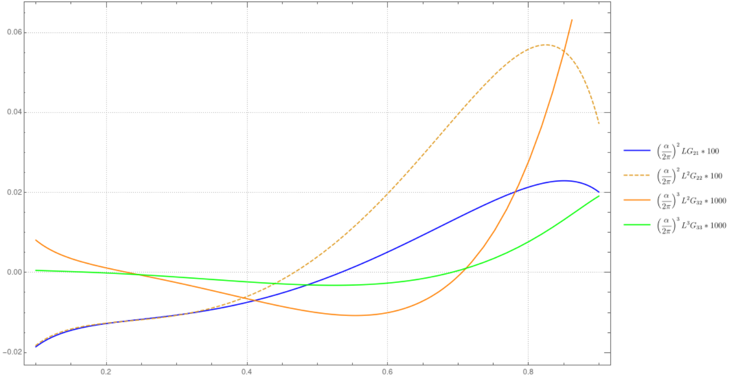
$$G(x) = \left(g_e^{(0)}(z) + \frac{\alpha}{2\pi} g_e^{(1)}(z)\right) \otimes \left[D_{ee}^{(\mathrm{III})}\right]_T + \left(g_\gamma^{(0)}(z) + \frac{\alpha}{2\pi} g_\gamma^{(1)}(z)\right) \otimes \left[D_{e\gamma}^{(\mathrm{II})}\right]_T$$

$$\begin{aligned}
&g_e^{(0)}(z) = z^2(1-2z),\\
&g_e^{(1)}(z) = 2z^2(1-2z)\left(\ln(1-z)^2 - 4\mathrm{Li}_2(1-z) - \ln(z)\ln(1-z) - 2\ln(z)^2\right)\\
&+\left(\frac{11}{3} - \frac{4}{3z} - 6z - \frac{17}{3}z^2 + \frac{34}{3}z^3\right)\ln(1-z) + \left(-\frac{1}{3} - 6z^2 - 6z^3\right)\ln(z)\\
&-\frac{7}{6} + 3z + \frac{7}{6}z^2 + 3z^3,\\
&g_\gamma^{(0)}(z) = 0,\\
&g_\gamma^{(1)}(z) = \left(\frac{1}{3} - \frac{1}{3z} - \frac{2}{3}z^2 + \frac{2}{3}z^3\right)\ln(1-z) + \left(\frac{2}{3} - \frac{2}{3z}\right)\ln z - \frac{2}{3}\\
&+\frac{2}{3z} + \frac{11}{12}z - \frac{2}{3}z^2 - \frac{1}{4}z^3.
\end{aligned}$$

Corrections to muon decay spectrum to the order α^3



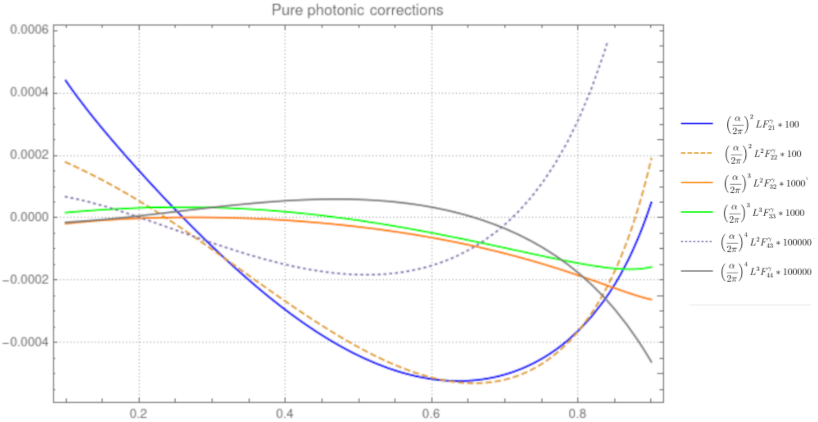
Corrections to muon decay spectrum to the order α^3



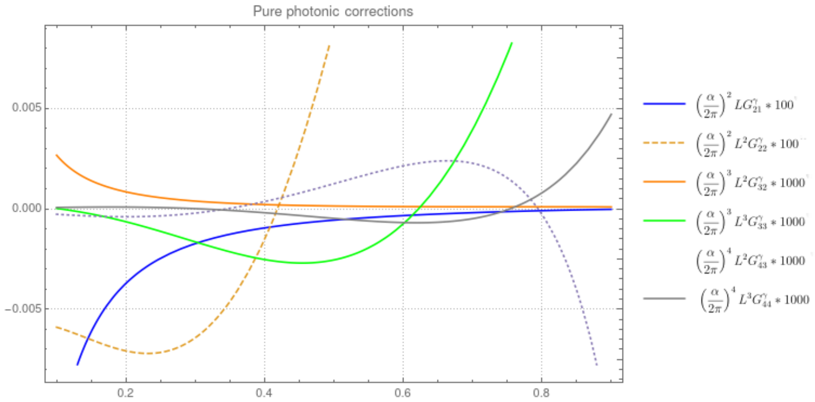
Conclusion

- Parton distribution functions approach allows to calculate logarithmic radiative corrections, which are numerically the most important
- Electron parton distribution functions are calculated by solving PDF evolution equations
- Application of the PDF approach to calculation of muon decay spectrum corrections is presented

Pure photonic corrections



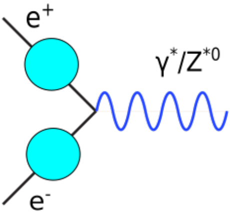
Pure photonic corrections



Splitting functions

e^+e^--annihilation

Consider initial state radiative corrections



$$\frac{d\sigma_{\bar{e}e}}{ds'} = \sigma^{(0)}\Big[D_{\bar{e}\bar{e}} \otimes D_{ee} \otimes \sigma_{\bar{e}e} + D_{\gamma e} \otimes D_{ee} \otimes \sigma_{e\gamma} + D_{ee} \otimes D_{e\bar{e}} \otimes \sigma_{ee}$$
$$+D_{\gamma e} \otimes D_{\bar{e}\bar{e}} \otimes \sigma_{\bar{e}\gamma} + D_{\gamma e} \otimes D_{\gamma\bar{e}} \otimes \sigma_{\gamma\gamma} + D_{\gamma e} \otimes D_{e\bar{e}} \otimes \sigma_{e\gamma} + D_{\bar{e}e} \otimes D_{\bar{e}\bar{e}} \otimes \sigma_{\bar{e}\bar{e}}$$
$$+D_{\bar{e}e} \otimes D_{\gamma\bar{e}} \otimes \sigma_{\bar{e}\gamma} + D_{\bar{e}e} \otimes D_{e\bar{e}} \otimes \sigma_{\bar{e}e}\Big]$$