Generalized Bargmann-Wigner construction for massive and massless relativistic fields

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Plan

Unitary Irreps of $ISO^{\uparrow}(1,3)$ (or its covering $ISL(2,\mathbb{C})$) $\longleftrightarrow$ relativistic particles

Classification of unitary irreps of Poincaré group $ISO^{\uparrow}(1,3)$, and its covering $ISL(2,\mathbb{C})$, was given by $\left[\begin{array}{c} E.Wigner\,(1939,1947), \\ V.Bargmann, E.Wigner\,(1948) \end{array}\right]$. Note that unitary irreps of the noncompact group $ISL(2,\mathbb{C})$, which are interesting from the physical point of view, are infinite dimensional. Irreducible representations of covering of the Poincaré group $ISL(2,\mathbb{C})$ are defined (as induced irreps) in the infinite dimensional space of the Wigner-Bargmann wave functions, which do not carry any information about the relativistic equations for the relativistic fields (e.g., about Dirac equations for spin $1/2$ fields, or Rarita-Schwinger equations for spin $3/2$ fields). The transformation of the Wigner-Bargmann wave functions into the local relativistic fields (corresponding to the irreps of the Poincaré group) are carried out, by definition, by the special Wigner operators. In this report we introduce and study generalized Wigner operators for massive and massless irreps of $ISL(2,\mathbb{C})$.

WB wave functions $\phi_{\bar{\alpha}}(k) \overset{\text{Wigner operators}}{\longleftrightarrow}$ relativistic fields $\psi_{\bar{\alpha}}(k)$

Here we show how the relativistic equations for local massive relativistic fields are dictated by the form of Wigner operators.

For the massless irreps of the infinite spin (or continue spin) the fields are parameterized by an additional variable – commuting vector or spinor. These variables are inf. dim. analogs of the discrete indices of compact subgroup $SU(2)$ in the massive case. The equations of motion for fields of infinite spin were derived in both formulations (with additional commuting vector, or spinor). We show that the relativistic fields for the standard massless helicity representations (constructed in this way) are obtained in the special limit of the infinite spin (or continue spin) irreps. The corresponding relativistic fields are gauge potentials and satisfy the relations that determine free massless higher spin fields.

To characterize the unitary irreps of d-dimensional Poincaré group $ISO^{\uparrow}(1, d-1)$, or its covering $\mathrm{ISpin}^{\uparrow}(1, d-1)$, we need to consider the corresponding irreps of the Lie algebra $\mathfrak{iso}(1, d-1) = i\mathrm{spin}(1, d-1)$ with generators $\hat{P}_n$ (components of momentum) and $\hat{M}^{mk}$ (components of the angular momentum) which satisfy defining relations

$$[\hat{P}_n, \hat{P}_m] = 0 \;, \quad [\hat{P}_n, \hat{M}_{mk}] = i\,(\eta_{kn}\hat{P}_m - \eta_{mn}\hat{P}_k) \;,$$

$$[\hat{M}_{nm}, \hat{M}_{k\ell}] = i(\eta_{nk}\hat{M}_{m\ell} - \eta_{mk}\hat{M}_{n\ell} + \eta_{m\ell}\hat{M}_{nk} - \eta_{n\ell}\hat{M}_{mk}) \;,$$

where $||\eta_{mk}|| = \mathrm{diag}(+1, -1, \ldots, -1)$ – metric in $\mathbb{R}^{1,d-1}$.

The Lie algebra $\mathfrak{iso}(1, d-1)$ has $[(d+1)/2]$ Casimir operators since the algebra $iso(1, d-1)$ is obtained by the contraction from the simple Lie algebra $so(d+1, \mathbb{C})$ of rank $[(d+1)/2]$, where $[q]$ denote the integer part of q.

Thus, the Lie algebra $\mathfrak{iso}(1,3)$ of the Poincaré group for d=4 has $[5/2] = 2$ Casimir operators.

The d-dim. Poincaré algebra $iso(1,3)$ has two Casimir operators

$$C_2 = \hat{P}^n \hat{P}_n \,, \quad C_4 = \hat{W}^n \hat{W}_n$$

where $\hat{W}_n = \frac{1}{2}\, \varepsilon_{nmkr}\, \hat{M}^{mk}\, \hat{P}^r$ are components of Pauli-Lubansky vector which satisfy

$$\hat{W}_n \hat{P}^n = 0 \,, \quad [\hat{W}_k ,\, \hat{P}_n] = 0 \,, \quad [\hat{W}_m ,\, \hat{W}_n] = i\, \varepsilon_{mnkr} \hat{W}^k\, \hat{P}^r \,.$$

Classification of the $ISL(2,\mathbb{C})$-irreps:
[E.P.Wigner (1939); V.Bargmann, E.P.Wigner(1948)]

1. Massive irreps. On the space of states of massive irreps the Casimir operators are proportional to the unite operator I:

$$\hat{P}^n \hat{P}_n = \mathrm{m}^2\, I \quad (\mathrm{m}^2 > 0), \qquad \hat{W}^n \hat{W}_n = -\mathrm{m}^2 j(j+1)\, I \,,$$

where the real number $\mathrm{m} > 0$ is called mass and the real number $j \in \mathbb{Z}_{\geq 0}/2$ is called spin.

2. Massless irreps. The Casimir operators of $iso(1,3)$ are

$$\hat{P}^n \hat{P}_n = \mathrm{m}^2 = 0 \,, \quad \hat{W}^2 = \hat{W}^n \hat{W}_n = -\mu^2 \,.$$

In this case we have two subcases: A. $\mu^2 = 0$ and B. $\mu^2 \neq 0$.

In massless case A, when $\mu^2 = 0$, we obtain

$$\hat{W}^2 = 0, \quad \hat{P}^2 = 0, \quad \hat{P}_n \hat{W}^n = 0 \ \overset{\mathbb{R}^{1,3}}{\Longrightarrow} \ \hat{W}_n = \hat{\Lambda} \cdot \hat{P}_n \ ,$$

where element $\hat{\Lambda} \in iso(1,3)$ is central and called helicity operator. Its eigenvalues are $\Lambda = 0, \pm 1/2, \pm 1, \pm 3/2, \dots$.
We call these representations as helicity representations.
For example, photon is a particle with two possible states characterized by helicities $\Lambda = \pm 1$.

In massless case B, when $\mu^2 \neq 0$, we have

$$\hat{W}^2 = -\mu^2, \quad \hat{P}^2 = 0, \quad \hat{P}_n \hat{W}^n = 0 \ .$$

These irreps are called the infinite spin (or continues spin) representations.

To fix the notation, we recall the definition of the covering group $ISL(2,\mathbb{C})$ of the Poincare group $ISO^{\uparrow}(1,3)$. The group $ISL(2,\mathbb{C})$ is the set of all pairs (A,X), where $A \in SL(2,\mathbb{C})$, and X is a Hermitian (2×2) matrix (i.e. belongs to the space **H** of Hermitian matrices), which can always be represented in the form ($x_m \in \mathbb{R}$)

$$X = x_0\,\sigma^0 + x_1\,\sigma^1 + x_2\,\sigma^2 + x_3\,\sigma^3 = (x\,\sigma) = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \in \mathbf{H}\,.$$

Here the components of the 4-vector: $\sigma^m = (\sigma^0 = I_2, \sigma^1, \sigma^2, \sigma^3)$, where $\sigma^k|_{k=1,2,3}$ are Pauli matrices, form the basis in $\mathbf{H} = \mathbb{R}^{\mathbf{1,3}}$. The multiplication in the group $ISL(2,\mathbb{C})$ is given by the formula

$$(A,Y)\cdot(A',X) = (A\cdot A',\ \underline{A\cdot X\cdot A^{\dagger}} + Y)\,.$$

From this formula we have the $SL(2,\mathbb{C})$ group action in the Minkowski space $\mathbf{H} = \mathbb{R}^{1,3}$

$$X \;\to\; X' = A\cdot X\cdot A^{\dagger}\ \in \mathbf{H} \quad \Rightarrow \quad (x'\sigma) = A\cdot(x\sigma)\cdot A^{\dagger}\,.$$

This action gives the Lorentz rotation of a vector $x \in \mathbb{R}^{1,3}$:

$$\sigma^k x'_k = x_m \, (A \cdot \sigma^m \cdot A^\dagger) = \sigma^k \, \Lambda_k{}^m(A) \, x_m \quad \Rightarrow \quad x'_k = \Lambda_k{}^m(A) \, x_m \ ,$$

where $X_{\alpha\dot\beta} = x_k \, \sigma^k_{\ \alpha\dot\beta}$, $(\alpha, \dot\beta = 1, 2)$ and the (4×4) matrix $||\Lambda^m{}_k(A)|| \in SO^\uparrow(1,3)$ is determined from the standard relations

$$\boxed{A \cdot \sigma^m \cdot A^\dagger = \sigma^k \, \Lambda_k{}^m(A)} \quad \Leftrightarrow \quad A_\xi{}^\alpha \, A^{*}_{\dot\gamma}{}^{\dot\beta} \, \sigma^m_{\alpha\dot\beta} = \sigma^k_{\xi\dot\gamma} \, \Lambda_k{}^m(A) \ ,$$

which we need below.

We also need to have dual set of σ-matrices:

$$\tilde\sigma^m = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3) \ , \qquad (\tilde\sigma^m)^{\dot\alpha\beta} \ .$$

In the massive case: $\mathrm{m} > 0$, the unitary irreps of the group $ISL(2,\mathbb{C})$ are characterized by spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ and act in the spaces of Wigner-Bargmann (WB) wave functions $\phi_{(\alpha_1 \dots \alpha_{2j})}(k)$, which are components of a completely symmetric $SU(2)$-tensor of rank $2j$:
[E.P.Wigner (1939,1947); V.Bargmann, E.P.Wigner(1948)]

$$[\mathcal{U}(A, x_m \sigma^m) \cdot \phi]_{\bar{\alpha}}(k) \equiv \phi'_{\bar{\alpha}}(k) = e^{ix^m k_m}\; T^{(j)}_{\bar{\alpha}\bar{\beta}}(h_{A,\Lambda^{-1}\cdot k})\; \phi_{\bar{\beta}}(\Lambda^{-1} \cdot k)\;.$$

Here $k = (k_0, k_1, k_2, k_3)$ denotes the four-momentum of a particle with mass m: $(k)^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2 = \mathrm{m}^2$, matrix $\Lambda \in SO^{\uparrow}(1,3)$ is defined by $A \in SL(2,\mathbb{C})$ as before

$$A \cdot \sigma^m \cdot A^{\dagger} = \sigma^k \; \Lambda_k{}^m(A)$$

and we use the concise notation $\phi_{\bar{\alpha}}(k) \equiv \phi_{(\alpha_1 \dots \alpha_{2j})}(k)$, the indices $\bar{\alpha}, \bar{\beta}$ are multi-indices $(\alpha_1 \dots \alpha_{2j})$, $(\beta_1 \dots \beta_{2j})$; $T^{(j)}$ is a finite-dimensional irreducible $SU(2)$ representation with spin j, acting in the space of symmetric spin-tensors $\phi_{(\alpha_1 \dots \alpha_{2j})}$ and $h_{A,\Lambda^{-1}\cdot k}$ - an element of stability subgroup $SU(2) \subset SL(2,\mathbb{C})$.

Stability subgroup

Let us fix some test momentum $q=(q_0,q_1,q_2,q_3)$ such that $(q)^2=\mathrm{m}^2$, $q_0>0$ (e.g., $q=(\mathrm{m},0,0,0)$) and choose a representative $A_{(k)}\in SL(2,\mathbb{C})$:

$$\boxed{(k\sigma)=A_{(k)}\cdot(q\sigma)\cdot A^{\dagger}_{(k)}} \;\Leftrightarrow\; k_m=(\Lambda_{(k)})_m^{\;\;n}\,q_n\;, \qquad (1)$$

where $(k\sigma)=k^n\sigma_n$, $(q\sigma)=q^n\sigma_n$. The relation between the matrices $A_{(k)}$ and $\Lambda_{(k)}\equiv\Lambda(A_{(k)})$ is the standard.

Definition. A *stability subgroup* (*little group*) $G_q\subset SL(2,\mathbb{C})$ of the test momentum q is the set of matrices $A\in SL(2,\mathbb{C})$ satisfying the condition

$$\boxed{A\cdot(q\sigma)\cdot A^{\dagger}=(q\sigma)} \;\Leftrightarrow\; A_{\alpha}^{\;\;\gamma}\,(q^n\sigma_n)_{\gamma\dot{\alpha}}\,(A^*)_{\dot{\gamma}}^{\;\;\dot{\alpha}}=(q^n\sigma_n)_{\alpha\dot{\gamma}}\;.$$

In the massive case $(q)^2=\mathrm{m}^2$, the stability subgroup G_q is isomorphic to $SU(2)$. It is evident for $q=(\mathrm{m},0,0,0)$, since $(q\sigma)=\mathrm{m}\cdot I_2$, but it is true for any choice of test momenta q.

The matrix $A_{(k)}\in SL(2,\mathbb{C})$ is defined up to right multiplication $A_{(k)}\to A_{(k)}\cdot U$ by an element $U\in G_q=SU(2)$:

$$(A_{(k)}\cdot U)\cdot(q\sigma)\cdot(A_{(k)}\cdot U)^{\dagger}=A_{(k)}\cdot(U\cdot(q\sigma)\cdot U^{\dagger})\cdot A^{\dagger}_{(k)}=(k\sigma)\;.$$

For each k we fix a unique matrix $A_{(k)}$.

Thus, unique representative element $A_{(k)} \in SL(2,\mathbb{C})$ numerates the left coset in $SL(2,\mathbb{C})$ with respect to the right action of the subgroup $G_q = SU(2)$ on the elements $A \in SL(2,\mathbb{C})$: $A \;\sim\; A \cdot U$, i.e. $A_{(k)}$ are points in the coset space $SL(2,\mathbb{C})/SU(2)$.

Now the elements $h_{A,\,k}$ of the stability subgroup $SU(2) \subset SL(2,\mathbb{C})$, which appeared in the definition of the E.Wigner massive irreps, are defined as

$$A \cdot A_{(k)} = A_{(\Lambda \cdot k)} \cdot h_{A,\,k} \quad \Rightarrow \quad A^{-1}_{(\Lambda \cdot k)} \cdot A \cdot A_{(k)} = h_{A,\,k} \in SU(2) \; ,$$

The first relation follows from

$$(A \cdot A_{(k)})\,(q\sigma)\,(A \cdot A_{(k)})^{\dagger} = A\,(k\sigma)\,A^{\dagger} = (\Lambda \cdot k, \sigma) = A_{(\Lambda \cdot k)}\,(q\sigma)\,A^{\dagger}_{(\Lambda \cdot k)} \; .$$

Note, that the matrices $(A_{(k)})_{\alpha}{}^{a}$ have two kind of indices: the left $SL(2,\mathbb{C})$-type index α and the right $SU(2)$-type index a.

Recall again the Wigner formula for unitary spin j irreps $\mathcal{U}$ of $ISL(2,\mathbb{C})$ defined by the following action of the element $(A,a) \in ISL(2,\mathbb{C})$:

$$[\mathcal{U}(A,a)\cdot\phi]_{\bar{\alpha}}(k) \equiv \phi'_{\bar{\alpha}}(k) = e^{ia^m k_m}\; T^{(j)}_{\bar{\alpha}\bar{\beta}}(h_{A,\Lambda^{-1}\cdot k})\;\phi_{\bar{\beta}}(\Lambda^{-1}\cdot k)\;.$$

Here (in massive case) $T^{(j)}$ – finite-dimensional irrep of $SU(2)$; we use the concise notation for WB wave function $\phi_{\bar{\alpha}}(k) \equiv \phi_{(\alpha_1\ldots\alpha_{2j})}(k)$. The element (dependent on k)

$$h_{A,\Lambda^{-1}\cdot k} = A^{-1}_{(k)}\cdot A\cdot A_{(\Lambda^{-1}\cdot k)} \;\;\in\;\; SU(2)\;, \qquad (1)$$

belongs to the stability subgroup $SU(2) \subset SL(2,\mathbb{C})$ and the matrix $\Lambda \in SO^{\uparrow}(1,3)$ is related to $A \in SL(2,\mathbb{C})$ in standard way.
Since $h_{A,\Lambda^{-1}\cdot k} \in SU(2)$, Eq. (1) is written in another equivalent form

$$h_{A,\Lambda^{-1}\cdot k} = h^{\dagger-1}_{A,\Lambda^{-1}\cdot k} = A^{\dagger}_{(k)}\cdot A^{\dagger-1}\cdot A^{\dagger-1}_{(\Lambda^{-1}\cdot k)}\;. \qquad (2)$$

Since $h_{A,\Lambda^{-1}\cdot k}$ depends on k the coordinate representation of $\phi_{\bar{\alpha}}(k)$ can not be a local field!!! How can we solve this problem?

In the representation $T^{(j)}$, the element $h \in SU(2)$ is the matrix, which can be written in the factorized form ($p + r = 2j$)

$$T^{(j)}_{\bar\beta\bar\alpha}(h) = \left(h^{\otimes(p+r)}\right)_{\bar\beta\bar\alpha} = \left[h_{\beta_1}{}^{\alpha_1}\cdots h_{\beta_p}{}^{\alpha_p}\cdot h_{\beta_{p+1}}{}^{\alpha_{p+1}}\cdots h_{\beta_{p+r}}{}^{\alpha_{p+r}}\right] =$$
$$= \left[h_{\beta_1}{}^{\alpha_1}\cdots h_{\beta_p}{}^{\alpha_p}\cdot \left(h^{\dagger-1}\right)_{\beta_{p+1}}{}^{\alpha_{p+1}}\cdots \left(h^{\dagger-1}\right)_{\beta_{p+r}}{}^{\alpha_{p+r}}\right],$$

where r factors are chosen as $h \rightarrow h^{\dagger-1}$, since $h \in SU(2)$.
Then, we use the factorized forms (1), (2) and write the matrix $T^{(j)}_{\bar\beta\bar\alpha}(h_{A,\Lambda^{-1}\cdot k})$ in the factorized form

$$T^{(j)}(h_{A,\Lambda^{-1}\cdot k}) = \left(A^{-1}_{(k)}\cdot A\cdot A_{(\Lambda^{-1}\cdot k)}\right)^{\otimes p}\otimes\left(A^{\dagger}_{(k)}\cdot A^{\dagger-1}\cdot A^{\dagger-1}_{(\Lambda^{-1}\cdot k)}\right)^{\otimes r} =$$
$$= \left(A^{-1\,\otimes p}_{(k)}\otimes A^{\dagger\,\otimes r}_{(k)}\right)\cdot\left(A^{\otimes p}\otimes A^{\dagger-1\,\otimes r}\right)\cdot\left(A^{\otimes p}_{(\Lambda^{-1}\cdot k)}\otimes A^{\dagger-1\,\otimes r}_{(\Lambda^{-1}\cdot k)}\right)$$

and for the Wigner formula we have $\phi'(k) = T^{(j)}(h_{A,\Lambda^{-1}\cdot k})\,\phi(\Lambda^{-1}\cdot k) =$

$$= \left(A^{\otimes p}_{(k)}\otimes A^{\dagger-1\,\otimes r}_{(k)}\right)^{-1}\cdot\left(A^{\otimes p}\otimes A^{\dagger-1\,\otimes r}\right)\cdot\underbrace{\left(A^{\otimes p}_{(\Lambda^{-1}\cdot k)}\otimes A^{\dagger-1\,\otimes r}_{(\Lambda^{-1}\cdot k)}\right)\phi(\Lambda^{-1}\cdot k)} =$$
$$= \left(A^{\otimes p}_{(k)}\otimes A^{\dagger-1\,\otimes r}_{(k)}\right)^{-1}\cdot\left(A^{\otimes p}\otimes A^{\dagger-1\,\otimes r}\right)\cdot\psi^{(r)}(\Lambda^{-1}\cdot k)$$

Here we introduce (instead of the Wigner WFs $\phi_{(\delta_1 \ldots \delta_{p+r})}(k)$) spin-tensor fields of $(\frac{p}{2}, \frac{r}{2})$-type (with r dotted and p undotted indices):

$$\psi^{(r)(\dot{\beta}_1 \ldots \dot{\beta}_r)}_{(\alpha_1 \ldots \alpha_p)}(k) := \Big[[A_{(k)}^{\otimes p} \otimes (A_{(k)}^{\dagger -1})^{\otimes r}] \cdot \phi(k) \Big]^{(\dot{\beta}_1 \ldots \dot{\beta}_r)}_{(\alpha_1 \ldots \alpha_p)} =$$

$$= (A_{(k)})_{\alpha_1 \ldots \alpha_p}^{\;\;\delta_1 \ldots \delta_p} \cdot (A_{(k)}^{-1 \dagger})^{\dot{\beta}_{p+1} \ldots \dot{\beta}_{p+r}; \delta_{p+1} \ldots \delta_{p+r}} \phi_{(\delta_1 \ldots \delta_p \delta_{p+1} \ldots \delta_{p+r})}(k) \; ,$$

where

$$(A_{(k)})_{\alpha_1 \ldots \alpha_p}^{\;\;\delta_1 \ldots \delta_p} = (A_{(k)})_{\alpha_1}^{\;\;\delta_1} \cdots (A_{(k)})_{\alpha_p}^{\;\;\delta_p} \; ,$$

$$\mathrm{m}^{-r} (A_{(k)}^{-1\dagger} \cdot (q\tilde{\sigma}))^{\dot{\beta}_1 \ldots \dot{\beta}_r; \delta_1 \ldots \delta_r} = \mathrm{m}^{-r} (A_{(k)}^{-1\dagger} \cdot (q\tilde{\sigma}))^{\dot{\beta}_1 \delta_1} \cdots (A_{(k)}^{-1\dagger} \cdot (q\tilde{\sigma}))^{\dot{\beta}_r \delta_r}$$

and we restore the case of the arbitrary test momentum q.

The upper index (r) of the spin-tensors $\psi^{(r)}$ distinguishes these spin-tensors with respect to the number of dotted indices.

Definition. The operators $A_{(k)}^{\otimes p} \otimes \left(\frac{1}{\mathrm{m}} A_{(k)}^{\dagger -1}(q\tilde{\sigma})\right)^{\otimes r}$ which convert Wigner wave functions $\phi(k)$ into spin-tensor fields $\psi^{(r)}(k)$ of $(\frac{p}{2}, \frac{r}{2})$-type are called *the Wigner operators*.

Proposition 1. The $ISL(2, \mathbb{C})$-representation $\mathcal{U}$ is written for fields $\psi^{(r)}$ as following

$$[\mathcal{U}(A, a) \cdot \psi^{(r)}]^{(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k) =$$

$$= e^{ia^m k_m} \left[A_{\alpha_1 \dots \alpha_p}^{\ \gamma_1 \dots \gamma_p} \ (A^{\dagger -1})^{\dot{\beta}_1 \dots \dot{\beta}_r}_{\ \dot{\kappa}_1 \dots \dot{\kappa}_r} \right] \ \psi^{(r)(\dot{\kappa}_1 \dots \dot{\kappa}_r)}_{(\gamma_1 \dots \gamma_p)}(\Lambda^{-1} \cdot k) \ ,$$

where $A^{\dots}_{\dots} (A^{\dagger -1})^{\dots}_{\dots} = [A^{\otimes p} \otimes (A^{\dagger -1})^{\otimes r}]^{\dots}_{\dots}$, $A \in SL(2, \mathbb{C})$.

Thus, the coordinate representation of the functions $\psi^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k)$ are the local relativistic fields.

Proposition 2. *The wave functions $\psi^{(r)}$ satisfy the Dirac-Pauli-Fierz (DPF) equations [P.A.M.Dirac (1936), M. Fierz and W. Pauli (1939)]:*

$$k^m (\tilde{\sigma}_m)^{\dot{\gamma}_1 \alpha_1} \psi^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k) = \mathrm{m}\, \psi^{(r+1)(\dot{\gamma}_1 \dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_2 \dots \alpha_p)}(k) \,, \quad (r = 0, \dots, 2j-1) \,,$$

$$k^m (\sigma_m)_{\gamma_1 \dot{\beta}_1} \psi^{(r)(\dot{\beta}_1 \dots \dot{\beta}_r)}_{(\alpha_1 \dots \alpha_p)}(k) = \mathrm{m}\, \psi^{(r-1)(\dot{\beta}_2 \dots \dot{\beta}_r)}_{(\gamma_1 \alpha_1 \dots \alpha_p)}(k) \,, \quad (r = 1, \dots, 2j) \,,$$

which describe the dynamics of a massive relativistic particle with spin $j = (p+r)/2$. The compatibility conditions for the system of DPF eqs are given by the mass shell relations $(k^n k_n - m^2)\, \psi^{(r)}(k) = 0$.

Proof. Use the definitions of matrices $A_{(k)} \in SL(2,\mathbb{C})/SU(2)$:

$$(k\tilde{\sigma}) \cdot A_{(k)} = A^{\dagger -1}_{(k)} \cdot (q\tilde{\sigma}) \,, \quad (k\sigma) \cdot A^{\dagger -1}_{(k)} = A_{(k)} \cdot (q\sigma) \,,$$

where the test momentum frame is $(q\tilde{\sigma}) = (q\sigma) = \mathrm{m}\, I_2$.

Example. For the case of spin $j = 1/2$ we have $(p+r) = 1$ and obtain Dirac eqs.

$$k^m (\tilde{\sigma}_m)^{\dot{\gamma}\alpha}\, \psi^{(0)}_{\alpha}(k) = \mathrm{m}\, \psi^{(1)\dot{\gamma}}(k) \,, \quad k^m (\sigma_m)_{\gamma\dot{\beta}}\, \psi^{(1)\dot{\beta}}(k) = \mathrm{m}\, \psi^{(0)}_{\gamma}(k) \,.$$

In the case of $p + r = 2j$, the system of spin-tensor wave functions $\psi^{(r)}$ which obey the Dirac-Pauli-Fierz equations describes relativistic particles with spin j.

Proposition 3. Spin-tensor wave functions $\psi^{(r)(\dot\beta_1 \dots \dot\beta_r)}_{(\alpha_1 \dots \alpha_p)}(k)$ of type $(\frac{p}{2}, \frac{r}{2})$, which obey the Dirac-Pauli-Fierz equations, automatically satisfy the equations

$$[(\hat{W}^m\, \hat{W}_m)\, \psi]^{(r)(\dot\beta_1 \dots \dot\beta_r)}_{(\alpha_1 \dots \alpha_p)}(k) = -\mathrm{m}^2 j(j+1)\, \psi^{(r)(\dot\beta_1 \dots \dot\beta_r)}_{(\alpha_1 \dots \alpha_p)}(k)\,,$$

where $j = (\frac{p}{2} + \frac{r}{2})$, $\hat{W}_m$ are the components of the Pauli-Lubanski vector

$$\hat{W}_m = \frac{1}{2} \varepsilon_{mnij} M^{ij} P^n = \frac{1}{2} \varepsilon_{mnij} \hat{\Sigma}^{ij} P^n\,,$$

and $\hat{W}_m \hat{W}^m$ is the Casimir operator for the group $ISL(2, \mathbb{C})$; $\hat{\Sigma}^{ij}$ — spin part of M^{ij}.

The matrices $A_{(k)}$ numerate points of the coset space $SL(2,\mathbb{C})/SU(2)$. The left action of the group $SL(2,\mathbb{C})$ on $SL(2,\mathbb{C})/SU(2)$ is

$$A \cdot A_{(k)} = A_{(\Lambda \cdot k)} \cdot U_{A,k} \; , \qquad A \in SL(2,\mathbb{C}) \; , \;\; \Lambda \in SO^{\uparrow}(1,3) \; ,$$

where matrices A and Λ are related by standard formula $A\vec{\sigma}A^{\dagger} = \Lambda\vec{\sigma}$ and the element $U_{A,k} \in SU(2)$ depends on A and momentum k. Under this left action the element $A \in SL(2,\mathbb{C})$ transforms two columns of the matrix $A_{(k)}$ as two Weyl spinors. Therefore, it is convenient to represent the matrix $A_{(k)}$ by using two Weyl spinors μ and λ with components $\mu_\alpha, \; \lambda_\alpha$ (the matrix $A^{\dagger}_{(k)}$ will be correspondingly expressed in terms of the conjugate spinors $\overline{\mu}$ and $\overline{\lambda}$) in the following way:

$$(A_{(k)})_{\alpha}{}^{\beta} = \frac{1}{(\mu^{\rho}\,\lambda_{\rho})^{1/2}} \begin{pmatrix} \mu_1 & \lambda_1 \\ \mu_2 & \lambda_2 \end{pmatrix} \;\Rightarrow\; (A^{\dagger -1}_{(k)})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{(\overline{\mu}^{\dot{\rho}}\,\overline{\lambda}_{\dot{\rho}})^{1/2}} \begin{pmatrix} \overline{\lambda}_{\dot{2}} & -\overline{\mu}_{\dot{2}} \\ -\overline{\lambda}_{\dot{1}} & \overline{\mu}_{\dot{1}} \end{pmatrix} ,$$

Massless irreps

For massless irreps we choose the test vector $\overset{\circ}{p} \in \mathbb{R}^{1,3}$ as follows

$$||\overset{\circ}{p}_\nu|| = (\overset{\circ}{p}_0, \overset{\circ}{p}_1, \overset{\circ}{p}_2, \overset{\circ}{p}_3) = (E, 0, 0, E) \tag{3}$$

By definition, the finite-dimensional Wigner operators are the matrices $A_{(p)} \in SL(2, \mathbb{C})$ that transform the test momentum $\overset{\circ}{p}$ into an arbitrary momentum p

$$A_{(p)}(\overset{\circ}{p}\,\sigma)A^\dagger_{(p)} = (p\,\sigma)\,, \tag{4}$$

where $(p\,\sigma) := p_\mu \sigma^\mu$. The stability subgroup $G_{\overset{\circ}{p}}$ of $\overset{\circ}{p}$ is formed by matrices $h \in SL(2, \mathbb{C})$ that preserve $\overset{\circ}{p}$:

$$h(\overset{\circ}{p}\,\sigma)h^\dagger = (\overset{\circ}{p}\,\sigma)\,, \tag{5}$$

Equation (5) defining the stability subgroup $G_{\overset{\circ}{p}}$ of the test momentum $\overset{\circ}{p}$ given in (3) has the following solution

$$h = \begin{pmatrix} e^{\frac{i}{2}\theta} & e^{-\frac{i}{2}\theta}\,\mathbf{b} \\ 0 & e^{-\frac{i}{2}\theta} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\theta} & 0 \\ 0 & e^{-\frac{i}{2}\theta} \end{pmatrix}\,, \tag{6}$$

where $\theta \in [0, 2\pi]$ and $\mathbf{b} = b_1 + ib_2$.

The matrices (6) form the $ISO(2)$ group, i.e. $G_{\overset{\circ}{p}} \cong ISO(2)$ and stability group is not compact. The generators of $ISO(2)$:

$$\hat{R} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{T}_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad \hat{T}_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

satisfy the defining relations for the real algebra $\mathfrak{iso}(2)$

$$[\hat{T}_1, \hat{T}_2] = 0\,, \quad [\hat{R}, \hat{T}_a] = i\varepsilon_{ad}\hat{T}_d\ . \tag{8}$$

An element $h(\theta, \vec{b})$ of the $ISO(2)$ group can be written as a product

$$h(\theta, \vec{b}) = T(\vec{b}) \cdot R(\theta) \equiv e^{-ib_a \hat{T}_a}\, e^{-i\theta \hat{R}}\,, \tag{9}$$

where $T(\vec{b})$ is the element of the translation subgroup of $ISO(2)$ and $R(\theta)$ is the element of $SO(2) \subset ISO(2)$. Here $\vec{b} = (b_1, b_2) \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$ is the angular variable.

The unitary irreps of $ISO(2)$ group is given by the relation

$$\Phi'(\varphi) \;=\; \Big[\mathcal{U}\big(h(\theta,\vec{b})\big)\,\Phi\Big](\varphi) \;=\; e^{-i\,\vec{b}\cdot\vec{t}_\varphi}\Phi(\varphi-\theta)\,. \qquad (10)$$

It is also convenient to use another discrete basis $|\,n\,\rangle$, $n \in \mathbb{Z}$, in the space of the unitary irrep of $ISO(2)$ group. In this basis, the generator $\hat{R}$ is diagonal while the generators of T_a are not diagonal

$$\hat{R}|\,n\,\rangle = n|\,n\,\rangle\,, \qquad T_{\pm}|\,n\,\rangle = \rho|\,n\pm 1\,\rangle\,. \qquad (11)$$

The set of vectors $|\,n\,\rangle$ are orthogonal and complete $\langle\,n\,|\,m\,\rangle = \delta_{nm}$, $\sum_{n=-\infty}^{\infty}|\,n\,\rangle\langle\,n\,| = 1$. The function $\langle\,\varphi\,|\,n\,\rangle$ relating basis vectors $|\,n\,\rangle$ and $|\,\varphi\,\rangle$ is $\langle\,\varphi\,|\,n\,\rangle = \frac{1}{\sqrt{2\pi}}\,e^{in\varphi}$. Thus the wave function $\Phi(\varphi)$ is expanded as a Fourier series ($\Phi_n := \langle\,n\,|\,\Phi\,\rangle/\sqrt{2\pi}$)

$$\Phi(\varphi) = \langle\,\varphi\,|\,\Phi\,\rangle = \sum_{n=-\infty}^{\infty}\langle\,\varphi\,|\,n\,\rangle\langle\,n\,|\,\Phi\,\rangle = \sum_{n=-\infty}^{\infty}\Phi_n e^{in\varphi}\,. \qquad (12)$$

The induced unitary representations of the group $SL(2,\mathbb{C})$ realized on the Wigner functions $\Phi(p,\varphi)$ are constructed according to (10) have the form:

$$\Phi'(p,\varphi) \;:=\; [U(A)\Phi](p,\varphi) \;=\; \sum_{\phi'} \mathcal{D}_{\varphi\varphi'}(\theta_{A,\Lambda^{-1}p}, \vec{b}_{A,\Lambda^{-1}p})\, \Phi(\Lambda^{-1}p,\varphi')\,,$$
$$= e^{-i\vec{b}_{A,\Lambda^{-1}p}\cdot\vec{t}_{\varphi}}\, \Phi(\Lambda^{-1}p, \varphi - \theta_{A,\Lambda^{-1}p})\,. \qquad (13)$$

The transformations $\mathcal{D}_{\varphi\varphi'}(\theta_{A,\Lambda^{-1}p}, \vec{b}_{A,\Lambda^{-1}p})$ of the Wigner functions depend on the momentum variable p_μ. We assume that the Lorentz-covariant field $\Psi(p,y)$, which describes massless particles, is constructed from WF $\Phi(p,\varphi)$ via integral transformation

$$\Psi(p,\eta) \;=\; \int_0^{2\pi} d\varphi\, \mathcal{A}(p,\eta,\varphi)\, \Phi(p,\varphi)\,, \qquad (14)$$

where $\eta = (\eta_0, \eta_1, \eta_2, \eta_3) \in \mathbb{R}^{1,3}$ is the set of auxiliary variables. The kernel $\mathcal{A}(p,\eta,\varphi)$ plays the role of the Wigner operator, which is an infinite-dimensional analogue of $A_{(p)}$ from (4).

In the kernel $\mathcal{A}(p,\eta,\varphi)$ the variables η and φ plays the role of the $SL(2,\mathbb{C})$-type and $ISO(2)$-type continues indices.

Let the relativistic field $\Psi(p,\eta)$ given in (14) be transformed under the action of the Lorentz group in the standard way:

$$\Psi'(p,\eta) \;=\; [U(A)\Psi](p,\eta) \;=\; \Psi(\Lambda^{-1}p,\Lambda^{-1}\eta)\,, \tag{15}$$

where the matrices A and Λ are related by standard way.

Knowing the explicit form of the unitary Lorentz transformation (13) of WF $\Phi(p,\varphi)$ and the corresponding transformation (15) of the field $\Psi(p,\eta)$, we find the equations that determine the kernel $\mathcal{A}(p,\eta,\varphi)$ of the Wigner operator.

As a result we obtain the following equation for the kernel $\mathcal{A}(p,\eta,\varphi)$:

$$\mathcal{A}(\Lambda^{-1}p,\Lambda^{-1}\eta,\varphi) \;=\; e^{-i\vec{b}_{A,\Lambda^{-1}p}\vec{t}_{\varphi+\theta_{A,\Lambda^{-1}p}}}\mathcal{A}(p,\eta,\varphi+\theta_{A,\Lambda^{-1}p})\,. \tag{16}$$

1. Non-singular solution

The expression for the kernel $\mathcal{A}(p,\eta,\varphi)$ is given by

$$\mathcal{A}(p,\eta,\varphi) = e^{i\mu\,\eta\cdot\varepsilon_{(1)}(\varphi)/(\eta\cdot p)}\, f(\eta\cdot\eta,\eta\cdot p)\,, \tag{17}$$

where we introduced the mass dimensional constant

$$\mu := E\rho\,, \tag{18}$$

$f((\eta)^2,\,\eta\cdot p)$ is an arbitrary function, and we also introduced two additional 4-vectors

$$\overset{\circ}{\varepsilon}_{(1)}(\varphi) := \overset{\circ}{\varepsilon}_{(1)}\cos\varphi - \overset{\circ}{\varepsilon}_{(2)}\sin\varphi\,, \qquad \overset{\circ}{\varepsilon}_{(2)}(\varphi) := \overset{\circ}{\varepsilon}_{(1)}\sin\varphi + \overset{\circ}{\varepsilon}_{(2)}\cos\varphi\,, \tag{19}$$

that are $SO(2)$-transformations of

$$(\overset{\circ}{\varepsilon}_{(1)})_\nu = (0,1,0,0)\,, \qquad (\overset{\circ}{\varepsilon}_{(2)})_\nu = (0,0,1,0)\,. \tag{20}$$

As a result, one obtains the relativistic field

$$\Psi(p,\eta) = \int_0^{2\pi} d\varphi\, e^{i\mu\,\eta\cdot\varepsilon_{(1)}(\varphi)/(\eta\cdot p)}\, f(\eta\cdot\eta,\eta\cdot p)\,\Phi(p,\varphi)\,. \tag{21}$$

2. Singular solution

$$\mathcal{A}(p,\eta,\varphi) \;=\; \delta(\eta\cdot p)\,\delta(\eta\cdot\varepsilon_{(2)}(\varphi))\,e^{i\mu\,\eta\cdot\varepsilon/(\eta\cdot\varepsilon_{(1)}(\varphi))}\,f(\eta\cdot\varepsilon_{(1)}(\varphi))\,, \qquad (22)$$

where we have introduced the vector

$$\varepsilon = \Lambda(A_{(p)})\,\overset{\circ}{\varepsilon}\,. \qquad (23)$$

The vector (23) is light-like and transverse to the vectors $\varepsilon_{(1)}(\varphi)$, $\varepsilon_{(2)}(\varphi)$:

$$\varepsilon\cdot\varepsilon = 0\,, \qquad \varepsilon\cdot\varepsilon_{(1)}(\varphi) = \varepsilon\cdot\varepsilon_{(2)}(\varphi) = 0\,. \qquad (24)$$

Moreover, it obeys the condition $\varepsilon\cdot p = 1$.
Expression (22) coincides with the generalized Wigner operator found in [P. Schuster, N. Toro, JHEP 09 (2013) 104, arXiv:1302.1198 [hep-th]; JHEP **09** (2013) 105, arXiv:1302.1577 [hep-th]].

The equation of motion of the field $\Psi(p,\eta)$ are

$$(\eta\cdot p)\,\Psi(p,\eta)=0\,. \tag{25}$$

$$\Big[i\sqrt{-(\eta\cdot\eta)}\,(p\cdot\frac{\partial}{\partial\eta})+\mu\Big]\,\Psi(p,\eta)=0\,. \tag{26}$$

Additional condition $f(\eta\cdot\varepsilon_{(1)}(\varphi))=\delta(\eta\cdot\varepsilon_{(1)}(\varphi)-1)$, fixing the function $f(\eta\cdot\varepsilon_{(1)}(\varphi))$, leads to the equation

$$\big[(\eta\cdot\eta)+1\big]\,\Psi(p,\eta)=0 \tag{27}$$

As a result, the equation (26) becomes:

$$\Big[i\,(p\cdot\frac{\partial}{\partial\eta})+\mu\Big]\,\Psi(p,\eta)=0\,. \tag{28}$$

Together with the massless condition $p^2\,\Psi(p,\eta)=0$, the equations (25), (27), (28) are the Bargmann-Wigner equations for infinite spin fields depending on an additional vector variable η. From these eqs. we have $\hat{W}^2\Psi(p,\eta)=-\mu^2\Psi(p,\eta)$.

the field $\Psi(p,u,\bar{u})$ is found from the Wigner wave function $\Phi(p,\varphi)$ in the form

$$\Psi(p,u,\bar{u}) \; = \; \int_0^{2\pi} d\varphi \, \mathcal{A}(p,u,\bar{u},\varphi) \, \Phi(p,\varphi) \,, \qquad (29)$$

where the generalized Wigner operator $\mathcal{A}(p,u,\bar{u},\varphi)$ maps a function of φ into a function depending on u^{α}, $\bar{u}^{\dot{\alpha}}$.

In this case we have equations

$$(\pi^{\alpha}u_{\alpha} - \sqrt{\mu})\,\Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,, \qquad (30)$$

$$(\bar{u}_{\dot{\alpha}}\bar{\pi}^{\dot{\alpha}} - \sqrt{\mu})\,\Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,. \qquad (31)$$

$$\left(\pi^{\beta}\frac{\partial}{\partial u^{\beta}} + i\sqrt{\mu}\right)\Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,, \qquad (32)$$

$$\left(\bar{\pi}^{\dot{\beta}}\frac{\partial}{\partial \bar{u}^{\dot{\beta}}} + i\sqrt{\mu}\right)\Psi(\pi,\bar{\pi},u,\bar{u}) \;=\; 0\,. \qquad (33)$$

Again from these eqs we deduce

$$\hat{W}^2\,\Psi(p,u,\bar{u}) = -\mu^2\,\Psi(p,u,\bar{u})\,, \qquad (34)$$

Helicity representations

Fourier expansion of $\Phi(p,\varphi)$:

$$\Phi(p,\varphi)=\sum_{n=-\infty}^{\infty}\Phi_n(p)e^{in\varphi}\,. \tag{35}$$

and the representation of $SL(2,\mathbb{C})$ becomes

$$[U(A)\Phi]_n(p)=\sum_{m=-\infty}^{\infty}\mathcal{D}_{nm}(\theta_{A,\Lambda^{-1}p},\vec{b}_{A,\Lambda^{-1}p})\,\Phi_m(\Lambda^{-1}p)\,, \tag{36}$$

where $\mathcal{D}_{nm}$ is the matrix of the little group element h in the discrete basis:

$$\mathcal{D}_{nm}(\theta,\vec{b})=(-ie^{i\beta})^{m-n}\,e^{-im\theta}J_{(m-n)}(b\rho)\,, \tag{37}$$

the $\beta,b\in\mathbb{R}$ are the polar coordinates of $\vec{b}=b\,(\cos\beta,\sin\beta)$ and $J_{(n)}(x)$ are the Bessel functions of integer order. In the case of $\rho\to 0$ we have $J_{n-m}(0)=\delta_{nm}$ and the matrix element (37) is written as

$$\mathcal{D}_{nm}(\theta,\vec{b})=\delta_{nm}e^{-in\theta}\,, \tag{38}$$

I.e. the matrix $\mathcal{D}(\theta,\vec{b})$ becomes diagonal and the transformation (36) is written as

$$[U(A)\Phi]_n(p) = e^{-in\theta}\,\Phi_n(\Lambda^{-1}p)\,. \tag{39}$$

The one-to-one correspondence of the relativistic fields $\Psi(p,\eta)$ and the Wigner wave functions is given by the integral transform

$$\Psi(p,\eta) \;=\; \int\limits_0^{2\pi} d\varphi\,\mathcal{A}(p,\eta,\varphi)\,\Phi(p,\varphi)\,, \tag{40}$$

where $\mathcal{A}(p,\eta,\varphi)$ – the generalized Wigner operator. In the discrete basis, the WF $\Phi_n(p)$ and the local fields $\Psi(p,\eta)$ are related as

$$\Psi(p,\eta) = \sum_{n=-\infty}^{\infty} \mathcal{A}(p,\eta,n)\Phi_n(p)\,, \tag{41}$$

where the kernel $\mathcal{A}(p,\eta,n)$ of the generalized Wigner operator is the Fourier component of $\mathcal{A}(p,\eta,\varphi)$:

$$\mathcal{A}(p,\eta,n) = \int\limits^{2\pi} d\varphi\,\mathcal{A}(p,\eta,\varphi)\,e^{in\varphi}\,. \tag{42}$$

Solving eqs for the kernel $\mathcal{A}(p,\eta,n)$, we find the explicit form of the generalized Wigner operator of helicity states for an arbitrary 4-momentum:

$$\mathcal{A}(p,\eta,n) = \begin{cases} \delta(\eta\cdot p)\,(\varepsilon_{(+)}\cdot\eta)^{n} & \text{with } n>0\,, \\ \delta(\eta\cdot p)\,(\varepsilon_{(-)}\cdot\eta)^{-n} & \text{with } n<0\,, \end{cases} \tag{43}$$

where the 4-polarization vectors $\varepsilon_{(\pm)}$ are used.
We use the kernel of the generalized Wigner operator (43) and define a relativistic field in the form

$$\Psi_n(p,\eta) \;=\; \delta(\eta\cdot p)\,F_n(p,\eta)\,, \tag{44}$$

where

$$F_n(p,\eta) \;=\; F_n^{(+)}(p,\eta) \;+\; F_n^{(-)}(p,\eta)\,, \qquad F_n^{(\pm)}(p,\eta) = (\varepsilon_{(\pm)}\cdot\eta)^n\,\Phi_{\pm n}(p)\,. \tag{45}$$

The component fields $F_n^{(+)}(p,\eta)$ and $F_n^{(-)}(p,\eta)$ describe states with positive and negative helicities $\lambda = n$ and $\lambda = -n$.

The explicit form of (45) reproduces the eqs for the fields $F_n(p, \eta)$:

$$p^2 F_n(p,\eta) = 0\,, \tag{46}$$

$$\left(p \cdot \frac{\partial}{\partial\eta}\right) F_n(p,\eta) = 0\,, \tag{47}$$

$$\left(\frac{\partial}{\partial\eta} \cdot \frac{\partial}{\partial\eta}\right) F_n(p,\eta) = 0\,, \tag{48}$$

$$\left(\eta \cdot \frac{\partial}{\partial\eta}\right) F_n(p,\eta) = n\, F_n(p,\eta)\,. \tag{49}$$

The last equation determines the degree of homogeneity for the field $F_n(p, \eta)$ in the variables η^μ. In addition, the presence in the definition of $\Psi_n(p, \eta)$ of the field $F_n(p, \eta)$ together with the δ-function $\delta(\eta \cdot p)$ leads to the following equivalence relation:

$$F_n(p,\eta) \sim F_n(p,\eta) + (p\cdot\eta)\,\epsilon_{n-1}(p,\eta)\,, \tag{50}$$

where the functions $\epsilon_{n-1}(p, \eta)$ satisfy equations (78) – (81).

Relation (50) is essentially a gauge transformation with parameters $\epsilon_{n-1}(p,\eta)$ and, therefore, the field $F_n(p,\eta)$ is a gauge field.
The standard tensor description of gauge fields is obtained after explicit selecting the polynomial dependence in η of the field $F_n(p,\eta)$:

$$F_n(p,\eta) = \eta^{\mu_1} \dots \eta^{\mu_n} f_{\mu_1 \dots \mu_n}(p) \tag{51}$$

and transferring to the coordinate representation. The corresponding coordinate tensor field $f_{\mu_1 \dots \mu_n}(x)$ is automatically totally symmetric $f_{\mu_1 \dots \mu_n}(x) = f_{(\mu_1 \dots \mu_n)}(x)$ and, thanks to (46)-(48), obeys the equations

$$\Box f_{\mu_1 \dots \mu_n}(x) = 0 \,, \qquad \partial^{\mu_1} f_{\mu_1 \dots \mu_n}(x) = 0 \,, \qquad \eta^{\mu_1 \mu_2} f_{\mu_1 \mu_2 \dots \mu_n}(x) = 0 \,. \tag{52}$$

In addition, the equivalence relation (50) means that the fields $f_{\mu_1 \dots \mu_n}(x)$ are defined up to the gauge transformations:

$$\delta f_{\mu_1 \mu_2 \dots \mu_n}(x) = \partial_{(\mu_1} \epsilon_{\mu_2 \dots \mu_n)}(x) \,. \tag{53}$$

Equations (52) and gauge symmetry (53) are standard conditions that define free massless higher spin fields.

In the case of zero helicity, the use of additional variables η^μ is not required. In this case, the relativistic field coincides with the Wigner wave function $\Psi_0(p) = \Phi_0(p)$, which is not a gauge field, and obeys only the Klein-Gordon equation in the momentum representation: $p^2\Psi_0(p) = 0$.

Summary and outlook

1.) In this report, on the basis of unitary representations of the covering group $ISL(2,\mathbb{C})$ of the Poincaré group, we have constructed explicit solutions of the wave equations for free massive particles of arbitrary spin j (the Dirac-Pauli-Fierz equations).Then we proposed the method for decomposing of these solutions into a sum over independent components corresponding to different polarizations.

2.) The most interesting examples corresponding to spins $j = 1/2, 1, 3/2$ and $j = 2$ were discussed in detail in [A.P.I., M.A.Podoinitsyn, Nucl. Phys. B929 (2018) 452].

3.) We have to stress that the massless case can also be considered in a similar manner. Just as in the massive case, the spin-tensor wave functions of free massless particles with arbitrary helicity are constructed from the vectors of spaces of the unitary massless Wigner representations for the covering group $ISL(2,\mathbb{C})$ of the Poincaré group.

4.) In the massless case the corresponding spin-tensor wave functions satisfy the Penrose equations (these equations for fields of massless particles were formulated by Penrose in the coordinate representation, instead of the Dirac-Pauli-Fierz equations. It is remarkable that instead of the two-spinor formalism, which is suitable for the massive case, we arrive in the massless case at the twistor formalism.