Net Baryon Number Probability Distribution as an Indicator of Phase Transition

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Outline

1. Roberge-Weiss approach.
2. Quantities under study:
   - net-baryon probability distributions $\mathcal{P}_n$ and $\mathbf{P}_n$;
   - their moments and cumulants;
   - their relation to the pressure and grand canonical partition function.
3. Evaluation of these quantities in lattice simulations.
4. Equation of State (EoS) $p = f(\rho)$ at $T > T_{RW}$ and $T < T_c$.
5. Asymptotic behavior of $\mathbf{P}_n$ at $n \to \infty$.
6. Phenomenological issues.

In collaboration aith V.A.Goy

Roberge-Weiss approach in QCD at $\mu_B \neq 0$:

Fock space includes only colorless states at all T and μ_B.

$$\theta \equiv \frac{\mu_B}{T} = \theta_R + \imath\theta_I$$

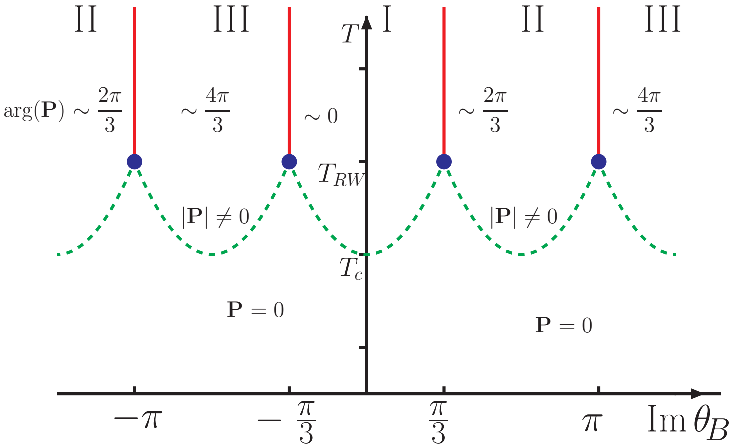
$$Z_{GC}(\theta_I) = Z_{GC}(\theta_I + 2\pi/N_c)$$

$\Downarrow$ $\Downarrow$

Quark number $\mathcal{Q}$ is a multiple of N_c

Grand canonical partition function

$$Z_{GC}(\theta, T, V) = \sum_j \langle j| \exp\left(\frac{-\hat{H} + \mu\hat{\mathcal{Q}}}{T}\right)|j\rangle$$


II
III
I
II
III
T
$\arg(\mathbf{P}) \sim \frac{2\pi}{3}$
$\sim \frac{4\pi}{3}$
~ 0
$\sim \frac{2\pi}{3}$
$\sim \frac{4\pi}{3}$
T_{RW}
$|\mathbf{P}| \neq 0$
$|\mathbf{P}| \neq 0$
T_c
$\mathbf{P} = 0$
$\mathbf{P} = 0$
$-\pi$
$-\frac{\pi}{3}$
$\frac{\pi}{3}$
π
$\operatorname{Im} \theta_B$

The quantities under study:

- probability $\mathcal{P}_n$ that the net baryon charge of the fireball at a given μ_B equals n
- and the corresponding
  - moments $\mu_k = \sum_{n=-\infty}^{+\infty} \mathcal{P}_n n^k$ and

    the respective moments generating function

    $$M(t) = 1 + \sum_{k=1}^{\infty} \frac{\mu_k}{k!}\, t^k$$
  - the cumulant generating function

    $$K(t) = \ln M(t) = \sum_{k=1}^{\infty} \frac{\varkappa_k}{k!}\, t^k$$

The probabilities $\mathcal{P}_n$ can be determined from

- experimental data

$$N_{events}\,(\text{Net-Baryon Number} = n) \ = \ = \ N_{events}\,(\text{Net-Proton Number} = 0.4n)$$

- lattice simulations
  (of the net-baryon density at imaginary μ_B)
- models of strong-interactiong matter
  — Hadron Resonance Gas (HRG) model —

Grand canonical partition function $Z_{GC}(\theta, T, V) \equiv Z_{GC}(\theta)$ can be expanded as follows:

$$Z_{GC}(\theta) \;=\; \exp\left(\frac{p(\theta)V}{T}\right) \;=\; \sum_{n=-\infty}^{\infty} Z_C(n)e^{n\theta},$$

The inverse transform:

$$Z_C(n) = \int_{-\pi}^{\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta)\bigg|_{\theta_R=0} \quad .$$

$$\theta = \mu_B/T = \theta_R + \imath\theta_I \;, \qquad \rho = \frac{1}{T}\frac{\partial p}{\partial \theta}$$

$$\mathcal{P}_n(\theta) = \frac{Z_C(n) e^{n\theta}}{Z_{GC}(\theta)}$$ - is the probability that

the baryon charge at the given T and μ_B equals n.

C-parity conservation implies $Z_C(n) = Z_C(-n)$

$$\Longrightarrow \qquad \frac{\mathcal{P}_n}{\mathcal{P}_{-n}} = \xi^{2n} \qquad \Longrightarrow \qquad \mu_B = \frac{T}{2n} \ln\left(\frac{\mathcal{P}_n}{\mathcal{P}_{-n}}\right)$$

- possible procedure of measurement of μ_B [A.Nakamura, K.Nagato 2013]
- criterion of thermodynamical equilibrium: μ_B measured for different n coincide

Net-baryon probability distribution at $\mu_B = 0$

$\mathbf{P}_n \equiv \mathcal{P}_n(\theta = 0)$ involve all info on θ-dependence:

$$\mathcal{P}_n(\theta) = \frac{Z_C(n) e^{n\theta}}{Z_{GC}(\theta)} = \mathbf{P}_n e^{n\theta} \frac{Z_{GC}(0)}{Z_{GC}(\theta)}$$

$$M_\theta(t) = \frac{Z_{GC}(t+\theta)}{Z_{GC}(\theta)} \longrightarrow \mathfrak{M}(t) = \frac{Z_{GC}(t)}{Z_{GC}(0)}$$

$$K_\theta(t) = \longrightarrow \mathfrak{K}(t) = \frac{\big(p(t) - p(0)\big) V}{T}$$

$$\mathbf{P}_n = \frac{Z_C(n)}{Z_{GC}(0)}$$

and the respective cumulants
in contrast to θ-dependent cumulants $\varkappa_n(\theta)$
coincide with the coefficients
of the Taylor expansion of the pressure in θ:

$$p(\theta) = p(0) + \sum_{n=1}^{\infty} \frac{\kappa_{2n}}{(2n)!}\theta^{2n}$$

Main attention is focused on

EXP.: $\varkappa_n(\theta)$ at small n instead of $\mathcal{P}_n(\theta)$

THEOR.: κ_n at small n instead of $\mathbf{P}_n(\theta)$ $\forall n$

because $\kappa_n = \varkappa_n(0)$

are related to the Taylor expansion of the pressure.

We argue that

Asymptotic behavior of $\mathbf{P}_n$ at $n \to \infty$

may become an indicator of the chiral phase transition

Problem: Given κ_n find $\mathbf{P}_n$

In lattice QCD at $\mathrm{Re}\mu_B = 0$, $\mathrm{Im}\mu_B \neq 0$
we employ the formula

$$Z_{GC}(\theta) = \int \mathbf{D}U e^{-S_G} (\det \mathcal{D}(\mu_B))^{N_f}$$

to find the net baryon number density ρ
and $\implies$ the grand canonical partition function

$$\rho(\theta) = \frac{1}{V} \frac{\partial (T \ln Z_{GC})}{\partial \mu_B} \implies$$

$$Z_{GC}(\theta_I)|_{\theta_R=0} = \exp\left(V \int_0^{\theta_I} \rho(x)\, dx \right)$$

Results of lattice simulations

$T = 1.35T_c > T_{RW}$: $\mathrm{Im}\rho(\theta_I)$ is $\frac{2\pi}{3}$-periodic function with discontinuities at $\theta_I = \frac{(2n+1)\pi}{3}$;

at $|\theta_I| < \frac{\pi}{3}$ is well fitted by the polynomial

$$\mathrm{Im}\rho(\theta_I) \simeq a_1\theta_I - a_3\theta_I^3$$

$T = 0.93T_c$: $\mathrm{Im}\rho(\theta_I)$ is well fitted by the sine

$$\mathrm{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I)$$

Results of lattice simulations

$T > T_{RW}$: $\mathrm{Im}\rho(\theta_I)$ is a periodic function fitted by the polynomial of the type

$$\mathrm{Im}\rho(\theta_I) \simeq a_1\theta_I - a_3\theta_I^3 + ... + \simeq a_n\theta_I^n$$

over each segment $\theta_I^{(n-1)} < \theta_I < \theta_I^{(n)}$,

where $\theta_I^{(n)} = \dfrac{(2n+1)\pi}{3}$;

$T \sim T_c$: $\mathrm{Im}\rho(\theta_I)$ should be fitted by

$$\mathrm{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I) + f_2 \sin(2\theta) + ... + f_n \sin(n\theta) + ...$$

where $\{f_n\}$ rapidly decreases with n.

Equation of State

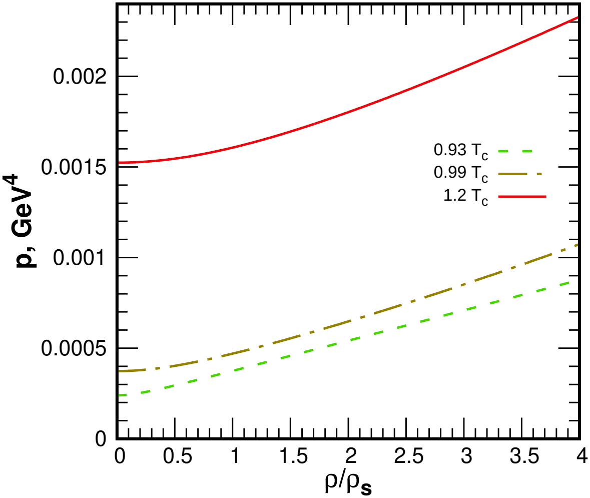
$T = 1.35T_c > T_{RW}$:

$$\frac{\rho}{T^3} = a_1\theta + a_3\theta^3$$
$$\frac{p}{T^4} = \frac{a_1}{2}\theta^2 + \frac{a_3}{4}\theta^4 + \hat{p}_0,$$

$T = 0.93T_c$:

$$\frac{p(\rho)}{T^4} = \hat{p}_0 + \left(\sqrt{\frac{\rho^2}{T^6} + f_1^2} - f_1\right) .$$

here $\hat{p}_0 = \left(\text{the pressure}/T^4\right)$ at $\theta = 0$.



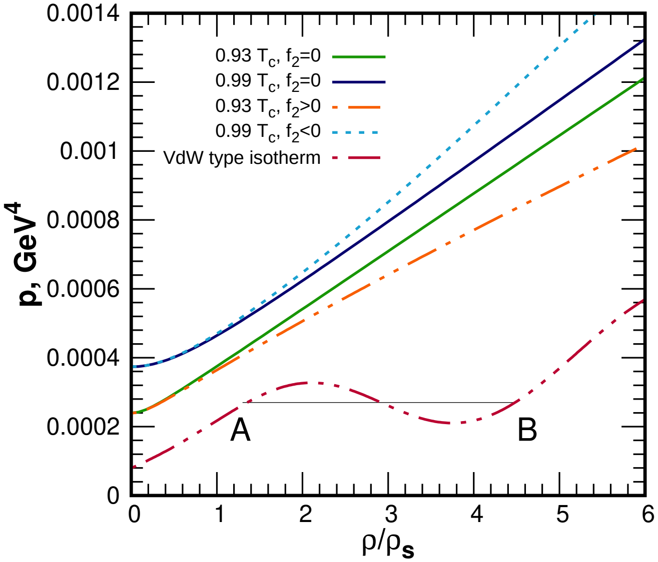
$\rho_s = \mathbf{0.153}$ fm^3; data for $\hat{p}_0$ are taken from HotQCD Collab.

Equation of State at $T \sim T_c$

$T = 0.99T_c$: $\mathrm{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I) + f_2 \sin(2\theta)$
$f_1 = 0.2541(8)\,, \quad f_2 = -0.0053(7)$

$$\hat{\rho} = f_1\, s + 2 f_2\, s\sqrt{s^2+1};$$
$$\hat{p} = f_1(\sqrt{s^2+1} - 1) + f_2\, s^2 + \hat{p}_0.$$

here $\hat{p}_0 = p/T^4$ at $\theta = 0$; $s = \sinh(\theta)$; $f_2 < 0$.


0.93 T_c, f_2=0
0.99 T_c, f_2=0
0.93 T_c, f_2>0
0.99 T_c, f_2<0
VdW type isotherm
A
B
p, GeV^4
ρ/ρ_s
0.0014
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
0
1
2
3
4
5
6

$T > T_{RW}$: $\mathbf{P}_n \simeq \exp\left(-\frac{n^2}{2a_1 VT^3}\right), \quad n \ll VT^3$

$$\mathbf{P}_n \simeq \exp\left(-\frac{3}{4}\sqrt[3]{\frac{3}{a_3}}\left(\frac{n}{VT^3}\right)^{4/3}\right), \quad \text{when } n \gg VT^3$$

$T < T_c$: coincidence with the HRG,

$$\mathbf{P}_n \simeq e^{-A} I_n(A)^{\dagger} \quad \Longrightarrow \quad A = 2\sqrt{b\bar{b}}$$

$(\bar{b})b$ is the average number of the (anti)baryons in the fireball

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† [Bornyakov et al., 1611.04229]

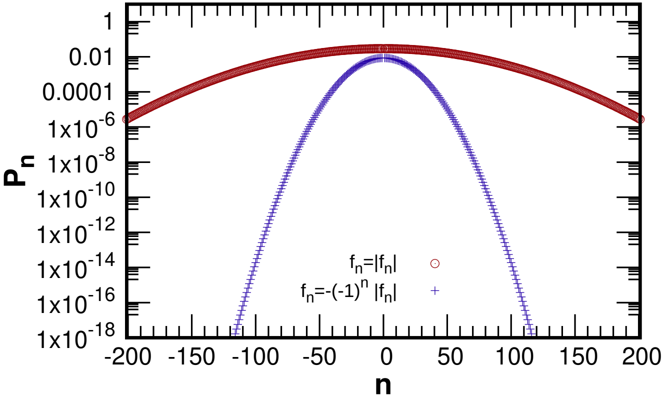
$$\mathrm{Im}\rho(\theta_I) \simeq a_1\theta_I + \ldots + a_{2J+1}\theta_I^{2J+1},$$

sign $a_{2J+1} = (-1)^J$

$$\mathbf{P}_n \sim \exp\left(-\frac{J}{J+1}\sqrt[J]{\frac{n^{J+1}}{\nu a_J}} \right) \qquad \nu = VT^3 \,.$$

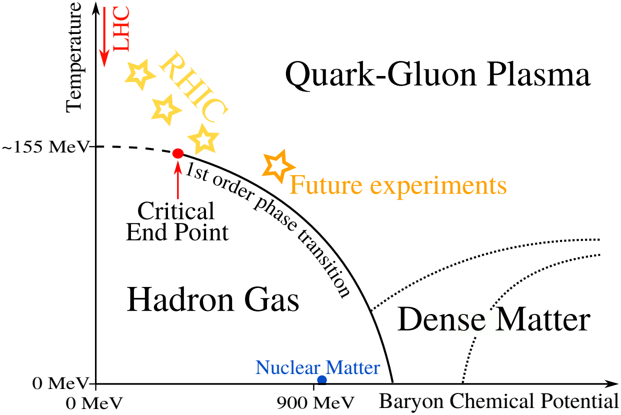
$$\mathrm{Im}\rho(\theta_I) \simeq f_1 \sin(\theta_I) \ldots + f_J \sin(J\theta), \quad f_J > 0 \; \forall J$$

$$\mathbf{P}_n \sim \frac{(\nu f_J)^{n/J}}{\Gamma\left(\frac{n}{J}+1\right)}, \qquad \nu = VT^3$$



1
0.01
0.0001
1x10-6
1x10-8
1x10-10
1x10-12
1x10-14
1x10-16
1x10-18
-200 -150 -100 -50 0 50 100 150 200
n
Pn
fn=|fn|
fn=-(-1)n |fn|

Hypothetical QCD phase diagram



Fireball evolution: $T_{ini}, \mu_B^{(ini)} \longrightarrow T_F, \mu_B^{(F)}$

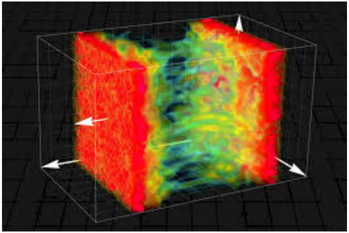
The Krein criterion states that the problem of moments becomes indeterminate when

$$\int dx \frac{\ln \varphi(x)}{(1+x^2)} > -\infty, \quad (1)$$

where $\varphi(x)$ is the probability density function.

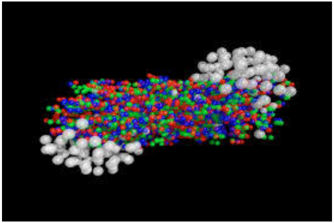
The rate of decrease in $\mathbf{P}_n$ at low temperatures is very close to the line of demarcation between probability mass functions generating determinate and indeterminate moment problems

Two scenarios of thermalization



1. Exchange of conserved charges (B, Q, S) proceeds during the fireball expansion.

Grand canonical approach works down to $T_{freezeout}$



2. The fireball after formation at an early stage is isolated from the remnants of colliding nuclei.

Evolution starts with the $Z_{GC}(\mu_{ini}, T, V)$ and proceeds with $Z_C(n, T, V)$.

Conclusions:

- Net-baryon number distribution $\mathbf{P}_n$ is evaluated on a lattice at $T > T_{RW}$ (it is similar to but doesn't coincide with the free theory) and at $T < T_c$ (coincides with the HRG predictions).
- Reconstruction of $\mathbf{P}_n$ from cumulants is either ambiguous or highly sensitive to small variations in higher-order cumulants. The analysis of experimental data based on the net-baryon number distribution involves additional information compared to that extracted from the set of cumulants.
- The dependence of the EoS on T and fit parameters has been used to formulate a possible scenario of emergence of the van der Waals isotherms corresponding to the first-order chiral phase transition.