Proof to count bound state nodes in supersymmetric quantum mechanics

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21st Lomonosov Conference on Elementary Particle Physics
August 24-30, 2023
Moscow State University

August 28, 2023

Sections

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Introduction

- New way of bound states nodes counting complimentary to ordinary program

- Classics:
  «Schrödinger equation»
  F.A.Berezin and M.A.Shubin proved the same statements by using the Sturm-Liouville theory in a set of theorems

- This talk:
  use instead the Supersymmetric QM math

Machinery of Supersymmetric Quantum Mechanics

- One-dimensional stationary Schrödinger equation

- A set of bound state levels ($\Psi_k^{(1)}$ and $\Psi_k^{(2)}$),
  Ψ_0 — the ground state to be real

- Define A^- linear differential operator by M.Crum:

$$A^- = -\frac{d}{dx} + \frac{\Psi_0'}{\Psi_0}$$

- Operators of super-generators Q and $\bar{Q} \equiv Q^\dagger$ by E.Witten:

$$Q = \frac{\hbar}{\sqrt{2m}} \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} \text{ and } \bar{Q} = \frac{\hbar}{\sqrt{2m}} \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \text{ act on } \begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \end{pmatrix}$$

Machinery of Supersymmetric Quantum Mechanics

- The supersymmetric Hamiltonian $\mathscr{H}$:

$$\mathscr{H} = \{Q, \bar{Q}\}, \qquad [Q, \mathscr{H}] = 0 \text{ and } Q^2 = \bar{Q}^2 = 0$$

- In matrix form:

$$\mathscr{H} = \frac{\hbar^2}{2m} \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \begin{pmatrix} H_1 - E_0 & 0 \\ 0 & H_2 - E_0 \end{pmatrix}$$

- So:

$$\hat{H}_1 \Psi_k^{(1)} = E_k \Psi_k^{(1)}, \qquad \hat{H}_2 \Psi_k^{(2)} = E_k \Psi_k^{(2)}$$

The ground state in discrete spectrum

- Ψ_0 satisfies the equation of ground state:

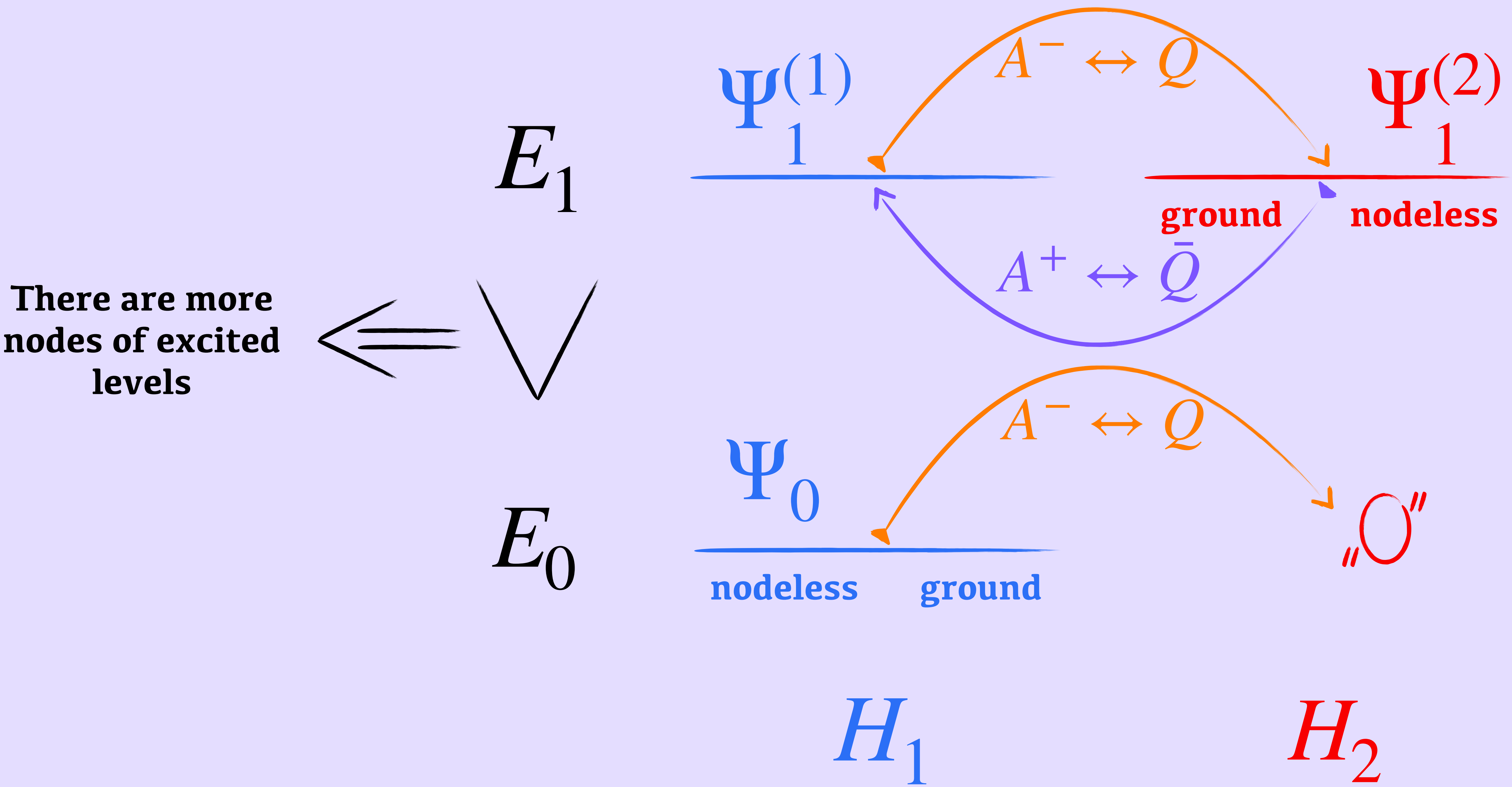
$$A^{-}\Psi_0 = 0$$

- Easy

$$A^{-}\Psi_0 = 0 \quad \Rightarrow \quad A^{-}|\Psi_0| = 0 \quad \Rightarrow \quad \Psi_0 = C|\Psi_0|$$

- Ψ_0 is nodeless

Counting the nodes: the scheme



Counting the nodes: in details

- Direct action:

$$[A^{+}\Psi_1^{(2)}]\cdot\Psi_0=\left[\frac{2m}{\hbar^2}(E_1-E_0)\Psi_1^{(1)}\right]\cdot\Psi_0=-W'_{\Psi_0\Psi_1^{(1)}},$$

where $W'_{\Psi_0\Psi_1^{(1)}}$ is Wronskian derivative of Ψ_0 and $\Psi_1^{(1)}$

- Explicitly:

$$[A^{+}\Psi_1^{(2)}]\cdot\Psi_0=(\Psi_0\cdot\Psi_1^{(2)})'$$

- While

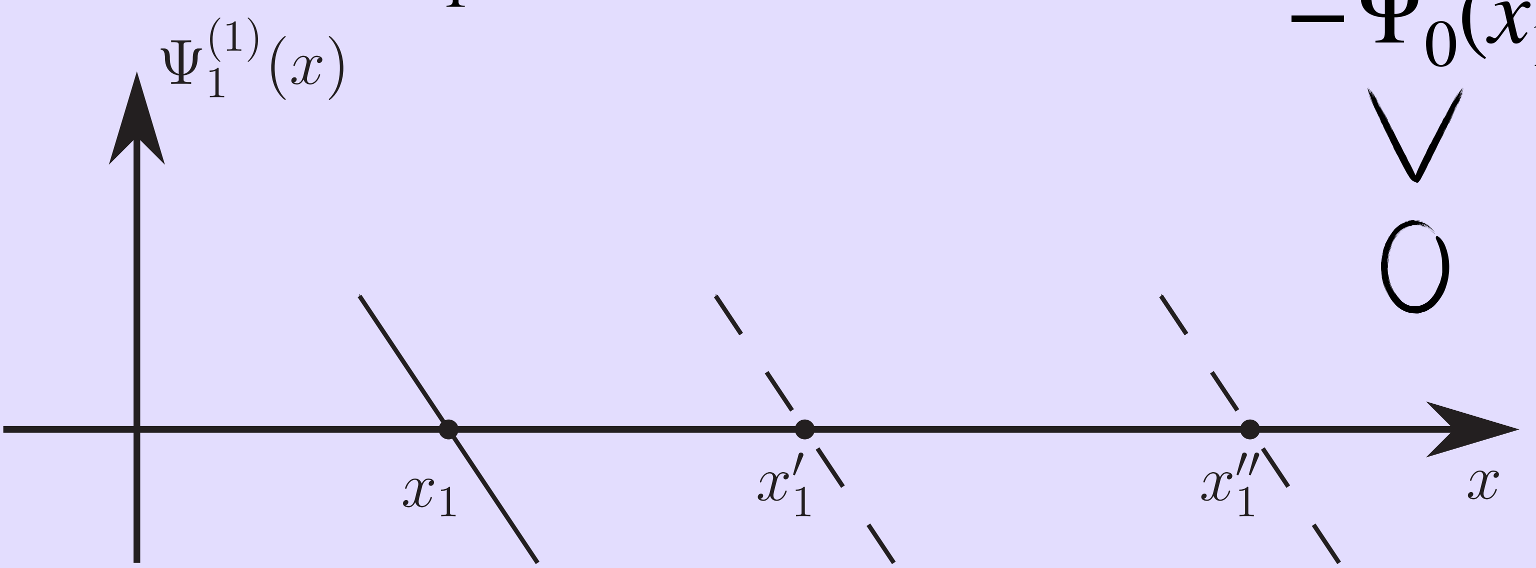
$$-W'_{\Psi_0\Psi_1^{(1)}}=(\Psi_0\cdot\Psi_1^{(2)})'\Rightarrow-W_{\Psi_0\Psi_1^{(1)}}=\Psi_0\cdot\Psi_1^{(2)}$$

- Revealing the Wronskian by it's definition:

$$\Psi_0'\underbrace{\Psi_1^{(1)}}_{\substack{\downarrow\\ E_1 > E_0 \Rightarrow \text{ more nodes } \exists \geqslant 1 \text{ node } \Rightarrow \exists -\infty < x_1 < +\infty \,:\, \Psi_1^{(1)}(x_1) = 0}} - \overbrace{\Psi_0}^{\substack{\text{No nodes inside}\\ \uparrow}}\Psi_1'^{(1)} = \Psi_0 \cdot \overbrace{\Psi_1^{(2)}}^{\substack{\text{No nodes inside}\\ \uparrow}}$$

- $\forall x_1 : \Psi_1^{(1)}(x_1) = 0$:

$$-\underset{\substack{\vee\\ 0}}{\Psi_0(x_1)} \cdot \Psi_1'^{(1)}(x_1) = \underset{\substack{\vee\\ 0}}{\Psi_0(x_1)} \cdot \underset{\substack{\vee\\ 0}}{\Psi_1^{(2)}(x_1)}$$



$\Downarrow$

continuous wave function

$\Psi_1'^{(1)}(x_1) < 0 \implies$ in all nodes derivative is negative $\implies \leqslant 1$ node $\implies$ exactly single node

Conclusion

- New modern proof for counting the nodes using the elegant math of Supersymmetric QM
- Implications $\longrightarrow$ could be used in QM books instead of classical Sturm-Liouville
- Researches about tetraquark and pentaquark states in QCD
- Detailed results are provided in arXiv preprint 2301.11303 (to be submitted)

Thank you for your attention!