Adler function and Bjorken polarized sum rule: PT expansions in powers of $SU(N_c)$ QCD β-function, $\{\beta\}$-expansion in QCD WITHOUT extra gluion and the conformal symmetry limit

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Plan

- The QCD studied quantities and two forms of the generalized Crewther relations (Broadhurst,Kataev(93)-Kataev,Mikhailov (12))
- The used form of the QCD expressions for the non-singlet parts of Adler D-function (and Bjorken polarized sum rule)
- The resulting expressions in the $\overline{MS}$-scheme
- The α_s^4 QCD expressions for the $\{\beta\}$-expanded coefficients (no gluino is needed)
- Difference with α_s^3 for the $\{\beta\}$-expansion in QCD+ gluino (Mikhailov(07), KM(12)-KM(15))
- Conformal symmetry limit identities are the same
- Problems for further studies: theory (for sure)- experiment (hopefully)

The basic definitions

$$R_{e^+e^-}(s) = \frac{1}{2\pi i} \int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{d_R D^(\sigma/\mu^2; a_s(\mu^2))}{\sigma} \, d\sigma \, \Bigg|_{\mu^2=s}$$

$$D\left(a_s(Q^2)\right) = \left(\sum_i q_i^2\right) D^{ns}\left(a_s(Q^2)\right) + \left(\sum_i q_i\right)^2 D^{si}\left(a_s(Q^2)\right)$$

The a_s^4 -term evaluated Baikov,Chetyrkin, Kuhn (2010);

$$S^{Bjp}(Q^2) = \int_0^1 [g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2)]dx =$$

$$\frac{g_A}{6} C_{ns}^{Bjp}(a_s(Q^2)) + \left(\sum_i q_i\right) c_4^{si} a_s^4(Q^2))$$

The a_s^4 term - BCK (2010)+ small si correction Larin (13)

The MS-scheme generalized Crewther relations

In the $\overline{MS}$-scheme the expansions read:

$$D^{ns}(a_s) = 1 + d_1 a_s + d_2 a_s^2 + d_3 a_s^3 + d_4 a_s^4$$
$$C_{ns}^{Bjp}(a_s) = 1 + c_1 a_s + c_2 a_s^2 + c_3 a_s^3 + c_4 a_s^4$$

The c_1 and d_1 depend from C_F, d_2, c_2 , d_3, c_3 depend from the monomials in C_F, C_A, $T_F n_f$, d_4, c_4 contain the contributions from d_F^{abcd}, d_A^{abcd} , i.e. symmetric tensors of the generators in the fundamental and adjoined representations. The generalized Crewther relation in the $\overline{MS}$-scheme reads

$$D^{ns}(a_s) C_{ns}^{Bjp}(a_s) = 1 + \Delta_{csb}(a_s)$$
$$\Delta_{csb}(a_s) = \left(\frac{\beta(a_s)}{a_s}\right) P(a_s) = \left(\frac{\beta(a_s)}{a_s}\right) \sum_{m \geq 1} K_m a_s^m$$

At a_s^3 discovered by Broadhurst,K(93), at a_s^4 confirmed by BChKuhn(10); **theoretical arguments** in orders Gabadadze,K(95);Crewther(97);Braun,Korchemsky,Muller(03)

The variant of the $\overline{MS}$-scheme generalized Crewther relation

The presented above $\overline{MS}$-scheme the CSB-term analytically known K_1 is proportional to C_F, K_2 contains C_F^2, $C_F C_A$ and $C_F T_F n_f$, K_3 contains C_F^3, $C_F^2 C_A$, $C_F C_A^2$, $C_F^2 T_F n_f$, $C_F C_A T_F n_f$ and $C_F (T_F n_f)^2$. However as shown by Kataev, Mikhailov (2012) to re-express it as (at least at the a_s^4-level)

$$
\Delta_{csb}(a_s) = \sum_{n\geq 1} \left(\frac{\beta(a_s)}{a_s} \right)^n P_n(a_s)
$$

$$
= \sum_{n\geq 1} \sum_{r\geq 1} \left(\frac{\beta(a_s)}{a_s} \right)^n P_n^{(r)}[k,m] C_F^k C_A^m a_s^r \quad .
$$

Here $r = k + m$ with $k \geq 1$ and $m \geq 0$, $P_n^{(r)}[k,m]$ contain rational numbers and transcendental Riemann ζ_{2l+1} numbers with $l \geq 1$. The $SU(N_c)$ monomials *do not* contain $T_F n_f$. In a_s^3 analogy appears in studies of the quantity related to static potential in QCD Grozin, Henn, Korchemsky, Marquard (2016)

New representations for the D^{ns}

Whether expansion in powers of conformal anomaly $\beta(a_s)/a_s$, where $\beta(a_s) = -\sum_{j\geq 0}\beta_j a_s^{j+2}$ is valid for the D^{ns}? Cvetic, Kataev (16): yes

$$D^{ns}(a_s) = 1 + \sum_{n=0}^{3}\left(\frac{\beta(a_s)}{a_s}\right)^n D_n(a_s)$$

$$D_n(a_s) = \sum_{r=1}^{4-n} a_s^r \sum_{k=1}^{r} D_n^{(r)}[k, r-k] C_F^k C_A^{r-k} + a_s^4 \delta_{n0} \times$$

$$\left(D_0^{(4)}[F,A]\frac{d_F^{abcd}d_A^{abcd}}{d_R} + D_0^{(4)}[F,F]\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right)$$

Why not to subdivide this $a_s^4 n_f$-dependent term as

$$\delta_{n0}D_0^{(4)}[F,F]n_f = \left(\delta_{n0}\frac{11C_A}{4T_F}D_0^{(4)}[F,F] + \delta_{n1}\frac{3}{T_F}D_1^{(4)}[F,F]\right)$$

with $D_0^{(4)}[F,F] = D_1^{(4)}[F,F]$? This contradicts QED limit- there is no such δ_{n1} contribution from light-by-light-type subgraph.

Expanding now β_0, β_1 , β_2 in terms of C_A, C_F, $T_f n_f$, using above presented representations and comparing them with the available analytical PT expansion for $D^{ns}(a_s)$ in $SU(N_c)$ up to a_s^4 with all colour structures fixed we obtain complete system of 22 equations. Solving it we get the expressions of D_0 (the polynomial prior $(\beta(a_s)/a_s)^0$) and D_1-D_3:

$$D_0(a_s) = \frac{3}{4}C_F a_s + \Big[-\frac{3}{32}C_F^2 + \frac{1}{16}C_F C_A\Big]a_s^2 + \Big[-\frac{69}{128}C_F^3 - \Big(\frac{101}{256} - \frac{33}{16}\zeta_3\Big)C_F^2 C_A - \Big(\frac{53}{192} + \frac{33}{16}\zeta_3\Big)C_F C_A^2\Big]a_s^3 + \Big[\Big(\frac{4157}{2048} + \frac{3}{8}\zeta_3\Big)C_F^4 - \Big(\frac{3509}{1536} + \frac{73}{128}\zeta_3 + \frac{165}{32}\zeta_5\Big)C_F^3 C_A + \Big(\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5\Big)C_F^2 C_A^2 - \Big(\frac{30863}{36864} + \frac{147}{128}\zeta_3 - \frac{165}{64}\zeta_5\Big)C_F C_A^3 + \Big(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\Big)\frac{d_F^{abcd}d_A^{abcd}}{d_R} - \Big(\frac{13}{16} + \zeta_3 - \frac{5}{2}\zeta_5\Big)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\Big)\Big]a_s^4$$

The expressions for D_1-D_3; in the $\overline{MS}$ (The results for $C_0(a_s)$-$C_4(a_s)$ were also obtained for the new representation of $C_{ns}^{Bjp}(a_s)$)

$$D_1(a_s) = \left(-\frac{33}{8} + 3\zeta_3\right) C_F a_s + \left[\left(\frac{111}{64} + 12\zeta_3 - 15\zeta_5\right) C_F^2 - \left(\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5\right) C_F C_A\right] a_s^2 + \left[\left(\frac{758}{128} + \frac{9}{16}\zeta_3 - \frac{165}{2}\zeta_5 + \frac{315}{4}\zeta_7\right) C_F^3 + \left(\frac{3737}{144} - \frac{3433}{64}\zeta_3 + \frac{99}{4}\zeta_3^2 + \frac{615}{16}\zeta_5 - \frac{315}{8}\zeta_7\right) C_F^2 C_A + \left(\frac{2695}{384} + \frac{1987}{64}\zeta_3 - \frac{99}{4}\zeta_3^2 - \frac{175}{32}\zeta_5 + \frac{105}{16}\zeta_7\right) C_F C_A^2\right] a_s^3,$$

$$D_2(a_s) = \left(\frac{151}{6} - 19\zeta_3\right) C_F a_s + \left[\left(-\frac{4159}{384} - \frac{2997}{16}\zeta_3 + 27\zeta_3^2 + \frac{375}{2}\zeta_5\right) C_F^2 + \left(\frac{14615}{256} + \frac{39}{16}\zeta_3 - \frac{9}{2}\zeta_3^2 - \frac{185}{4}\zeta_5\right) C_F C_A\right] a_s^2 \;,$$

$$D_3(a_s) = \left(-\frac{6131}{36} + \frac{203}{2}\zeta_3 + 45\zeta_5\right) C_F a_s \;.$$

The fixation of $\{\beta\}$-expanded coefficients for the D^{ns} at the a_s^4 level in $SU(N_c)$ **without** addition degrees of freedom (gluino)

Consider now the proposed by Mikhailov(07) $\{\beta\}$-expansion of PT coefficients for the Adler function:

$$d_1 = d_1[0] \quad , \quad d_2 = \beta_0 d_2[1] + d_2[0]$$
$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0] \ ,$$
$$d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1,1] + \beta_2 d_4[0,0,1]$$
$$+\beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 \, d_4[1] + d_4[0]$$

Without proposed by Kataev-Cvetic(17) application of the two-fold series expression in powers of the conformal anomaly and QCD coupling a_s it was possible to define the coefficients of the $\{\beta\}$-expressions for d_i and the similar expressions for the coefficients of Bjorken sum rule c_i at the a_s^3 level only in $SU(N_c)$ + multiplets of gluino (Mikhailov (07), Kataev, Mikhailov (12-15)).

The $\{\beta\}$ expanded terms for D^{ns} in $SU(N_c)$

We present these terms obtained in QCD in the $\overline{MS}$-scheme using the factorized representation, which is NOT proved, but **does not** contradict any theoretical results. They differ in part from obtained in QCD+gluino theory (Mikhailov (07))

$$d_1[0] = \frac{3}{4}C_F \;\; d_2[0] = \left(-\frac{3}{32}C_F^2 + \frac{1}{16}C_F C_A\right) \;\; d_2[1] = \left(\frac{33}{8} - 3\zeta_3\right)C_F$$

$$d_3[0] = -\frac{69}{128}C_F^3 - \left(\frac{101}{256} - \frac{33}{16}\zeta_3\right)C_F^2 C_A \neq +\mathbf{\frac{71}{64}C_F^2 C_A}$$

$$-\left(\frac{53}{192} + \frac{33}{16}\zeta_3\right)C_F C_A^2 \neq +\left(\mathbf{\frac{523}{768}} - \mathbf{\frac{27}{8}\zeta_3}\right)\mathbf{C_F C_A^2}$$

$$d_3[1] = \left(-\frac{111}{64} - 12\zeta_3 + 15\zeta_5\right)C_F^2 \neq \left(-\mathbf{\frac{27}{8}} - \mathbf{\frac{39}{4}\zeta_3} + \underline{\mathbf{15\zeta_5}}\right)\mathbf{C_F^2}$$

$$+\left(\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5\right)C_F C_A \neq \left(-\mathbf{\frac{9}{64}} + \mathbf{5\zeta_5} - \underline{\mathbf{\frac{5}{2}\zeta_5}}\right)\mathbf{C_F C_A}$$

$$d_3[0,1] = \left(\frac{33}{8} - 3\zeta_3\right)C_F \neq \left(\mathbf{\frac{101}{16}} - \mathbf{6\zeta_3}\right)\mathbf{C_F} \; d_3[2] = \left(\frac{151}{6} - 19\zeta_3\right)C_F$$

The $\{\beta\}$ expansion QCD expression for d_4 and c_4 were also obtained

To insure that this was done the $\overline{MS}$-expression for $d_4[0]$ is presented only.

$$d_4[0] = \Bigg[\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right)C_F^4 - \left(\frac{3509}{1536} + \frac{73}{128}\zeta_3 + \frac{165}{32}\zeta_5\right)C_F^3 C_A$$
$$+\left(\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5\right)C_F^2 C_A^2 - \left(\frac{30863}{36864} + \frac{147}{128}\zeta_3 - \frac{165}{64}\zeta_5\right)C_F C_A^3$$
$$+\left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} + \left(-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f$$

Note once more that $(d_F^{abcd}d_F^{abcd}/d_R)n_f$ term is not transformed to the QCD β_0-dependent structure, namely into $d_4[1]$, since in our basic aim is to have analogy with QED. In QED this gauge factor labels the contribution of the convergent light-by-light scattering insertion into photon vacuum polarization function and therefore in QED contribute to $d_4^{QED}[0]$-term.

The cross-check of the studies and the conformal symmetry limit identities

The multiplication of analytical expressions for the new QCD representations for $D^{ns}(a_s)$ and $C^{Bjp}_{ns}(a_s)$ at the a_s^4-level result in confirmation of the discovered by Kataev, Mikhailov (12) form of the $\overline{MS}$-scheme CSB expression in the generalized Crewther relation

$$\Delta_{csb}(a_s) = \sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

$$= \sum_{n\geq 1}\sum_{r\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n^{(r)}[k,m]C_F^k C_A^m a_s^r$$

with the same analytical coefficients. The QCD extension of the Crewther relation, obtained in the conformal symmetry limit with $\beta(a_s)\to 0$ now has the following form

$$(1 + D_0(a_s(Q^2)) \times (1 + C_0(a_s(Q^2)) = 1,$$

Notice cancellation of $d_F^{abcd}d_A^{abcd}/d_R$ and $d_F^{abcd}d_F^{abcd}/d_R)n_f$ a_s^4-terms. Extra indication of consistency of used

Model independent expressions

We study study the following from KS(15) and KM(15) links between the terms of $\{\beta\}$-expansion for $D^{ns}(a_s)$ and $C_{ns}^{Bjp}(a_s)$, which follow from the CS limit of generalized Crewther relation:

$$c_3[0] + d_3[0] = 2d_1 d_2[0] - d_1^3 = -\frac{9}{16}C_F^3 + \frac{3}{32}C_F^2 C_A,$$

$$c_4[0] + d_4[0] = 2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^4 =$$

$$-\frac{333}{1024}C_F^4 + \left(-\frac{363}{512} + \frac{99}{32}\zeta_3\right) C_F^3 C_A - \left(\frac{105}{256} + \frac{99}{32}\zeta_3\right) C_F^2 C_A^2,$$

$$c_2[1] + d_2[1] = c_3[0,1] + d_3[0,1] = c_4[0,0,1] + d_4[0,0,1]$$

$$= \left(\frac{21}{8} - 3\zeta_3\right)C_F,$$

$$c_3[1] + d_3[1] + d_1(c_2[1] - d_2[1]) = c_4[0,1] + d_4[0,1] + d_1(c_3[0,1]$$

$$-d_3[0,1]) = -\left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right)C_F^2 + \left(\frac{47}{48} - \zeta_3\right)C_F C_A \ .$$

They are scheme independent and valid in QCD with and without gluino.

Conclusion: A lot of ideas- not a lot of answers

Theoretical conclusions:

- In case of study of structure of PT series (say $\{\beta\}$-expansion partens and CS limit relations , which follow from generalized Crewther relation, it is possible to use either QCD or QCD+extra degrees of freedom
- In case we are interested in phenomenology of QCD gluionos are "artifacts". Better avoid them.
- New two-fold series representations for $D^{ns}(a_s)$ and $C_{ns}^{Bjp}(a_s)$ with ecat appearance of **powers of conformal anomaly** are proposed, They are true at a_s^4-level. Are they related to the generalized Crewther ?
- **Is it possible to prove these QCD expressions or understand them better?**

Continuation of possible studies

- We have only two series for RG-invariant functions. No other examples - can we check this structure using existing results ?
- Better understanding of the status of proposed by Brodsky et al PMC approach, links with other considerations and methods (say Cvetic,Valenzuela (2006); renormalon language
- Possible phenomenological applications-new studies of e^+e^- data (there are extra points from B-factory), new resummed studies of polarized Bjorken sum rule data- JLAB??
- Where CS Breaking terms may also manifest themselves in theory and (who knows - future phenomenology, say study of formfactors of decays of light mesons ??)

Waiting for your comments or input, ladies and gentlemen!!