Kapteyn type-II series' for synchrotron radiation problem

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Content:

Synchrotron radiation in even spacetime dimensions

Recurrence for LW potentials:

$$A^{\mu}_{D+2} = \frac{1}{D-1} \hat{\varrho}^{-1} \, \hat{d} \, A^{\mu}_{D}$$

Recurrence for the radiation prepotentials:

$$V^{\mu}_{D+2} = \frac{1}{D-1} \, \bar{d} \hat{\varrho}^{-1} V^{\mu}_{D}$$

Radiative EMT and radiative force:

$$T^{\mu\nu}_{\text{emit}} = V \cdot V \, \hat{c}^{\mu} \hat{c}^{\nu} \qquad f^{\mu}_{\text{emit}} = \int_{S^{D-2}} V^2_D \, \hat{c}^{\mu} \, \hat{\varrho}^{D-2} d\Omega$$

Synchrotron frequencies:

$$w_0 \equiv \frac{e_D H}{m} \qquad w \equiv \frac{e_D H}{m\gamma}$$

Intensivity:

$$I \equiv \frac{dE_{\text{emit}}}{dt} = -\frac{f^0_{\text{emit}}}{\gamma}\,,$$

Results:

$$(3+1):\quad I_4 = \frac{e_4^2 w_0^2}{6\pi}\,\gamma^2 v^2$$

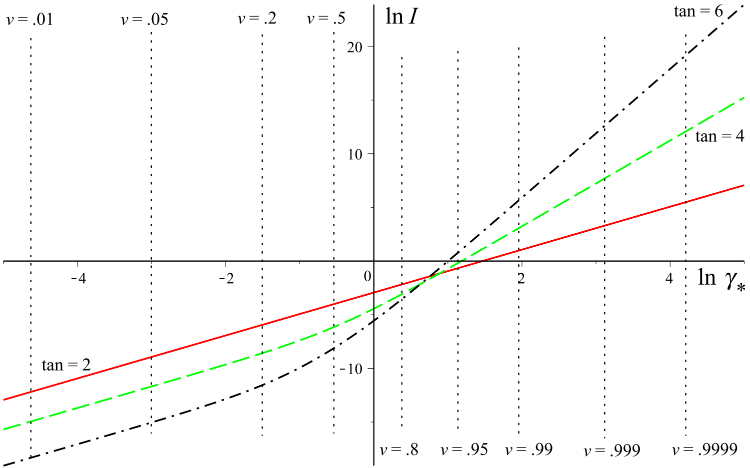
$$(5+1):\quad I_6 = \frac{e_6^2 w_0^4}{12\pi^2}\left(\gamma^2 v^2 + \frac{2}{5}\right)\gamma^2 v^2$$

$$(7+1):\quad I_8 = \frac{e_8^2 w_0^6}{72\pi^3}\left(5\gamma^4 v^4 + \frac{346}{105}\,\gamma^2 v^2 + \frac{9}{35}\right)\gamma^2 v^2$$

Leading power:

$$I_{2p} \sim e_{2p}^2 w_0^{2p-2} \sum_{k=1}^{p-1} (\gamma v)^{2k}$$

Doubly-logarithmic-mode plot ($\gamma_* = \gamma v$)



Plots of electromagnetic synchrotron radiation rate versus γ_* in doubly logarithmic mode for $D = 4$ (red, solid), $D = 6$ (green, dashed) and $D = 8$ (black, dashdotted), in units $e = w_0 = 1$

Discrete frequencies and wavevectors:

$$\omega_m = mw\,, \qquad k_m^\mu = wm(1,\boldsymbol{n})\,.$$

Radiation formulae:

$$\frac{d}{d\Omega}\begin{Bmatrix} I^{\rm sc} \\ I^{\rm em} \end{Bmatrix} = \frac{w^{D-2}}{2(2\pi)^{D-2}} \sum_{m=1}^{\infty} m^{D-2} \begin{Bmatrix} |j_m(\boldsymbol{k})|^2 \\ -j_m^*(\boldsymbol{k})\cdot j_m(\boldsymbol{k}) \end{Bmatrix},$$

where currents

$$\begin{Bmatrix} j_m(\boldsymbol{k}) \\ j_m^\mu(\boldsymbol{k}) \end{Bmatrix} \equiv \frac{w}{2\pi\gamma}\oint \begin{Bmatrix} q \\ e\dot{z}^\mu(t') \end{Bmatrix} \mathrm{e}^{i\boldsymbol{k}\boldsymbol{z}(t')}\,\mathrm{e}^{-iwmt'}\,dt'\,,$$

Polarisations (vector field):

$$\frac{dI^{\rm em}}{d\Omega} = \frac{w^{D-2}}{2(2\pi)^{D-2}} \sum_{m=1}^{\infty} m^{D-2} \sum_{\alpha=1}^{D-2} |j_\alpha^{(m)}|^2,$$

Special family of Kapteyn type-II series'

Symbolically

$$S^{D-2} = S^1 \cos\theta \times S^{D-4} \sin\theta$$

Angular distribution:

$$\frac{d}{d\theta} \begin{Bmatrix} I^{\rm sc} \\ I^{\rm em} \end{Bmatrix} = \zeta_D \sum_{m=1}^{\infty} m^{D-2} \begin{Bmatrix} q^2 \gamma^{-2} J_m^2(mv\cos\theta) \\ e^2 \left[v^2 {J'_m}^2(mv\cos\theta) + {\rm tg}^2\theta\, J_m^2(mv\cos\theta) \right] \end{Bmatrix}$$

where

$$\zeta_D = \frac{w^{D-2} \cos\theta\, \sin^{D-4}\theta}{(2\sqrt{\pi})^{D-3} \Gamma\left(\frac{D-3}{2}\right)}$$

Kapteyn series' ($z = v\cos\theta < 1$):

$$C_\lambda(z) = \sum_{m=1}^{\infty} m^\lambda J_m^2(mz)\,, \qquad D_\lambda(z) = \sum_{m=1}^{\infty} m^\lambda {J'_m}^2(mz)$$

Even index

$\lambda = 2$ [Schott'1912]:

$$C_2 = \frac{z^2(4+z^2)}{2^4(1-z^2)^{7/2}} \qquad D_2 = \frac{4+3z^2}{2^4(1-z^2)^{5/2}}$$

$\lambda = 4$ [Tautz, Lerche' 2007]:

$$C_4(z) = \frac{z^2(64+592z^2+472z^4+27z^6)}{2^8(1-z^2)^{13/2}}$$

Generic even $\lambda = 2s$ [Dominici' 2011]:

$$C_{2s}(z) = \frac{1}{2^{2s+1}} \left(\frac{\partial^{2s}}{\partial t^{2s}} \; {}_2F_1\left(s+1, s+\frac{1}{2}; 1; \frac{\mathrm{sh}^2 t}{t^2}\, z^2\right)\right)_{t=0}$$

Schematically,

$$C_{2s}(z) = \frac{z^2 P_{2s-1}(z^2)}{(1-z^2)^{3s+1/2}}, \qquad D_{2s}(z) = \frac{\tilde{P}_{2s-1}(z^2)}{(1-z^2)^{3s-1/2}}$$

[Marshall'1979]

$$G_\lambda(z) := \sum_{n=-\infty}^{\infty}{}' n^{\lambda-2}|n|\left(J_n'(nz) + \frac{\sqrt{1-z^2}}{z} J_n(nz)\right)^2$$

Recurrence for G:

$$G_{2s}(z) = \frac{1}{2z}\frac{d}{dz}\left(\frac{z^2}{\sqrt{1-z^2}} G_{2s-1}(z)\right)$$

Initial even-indexed G :

$$G_2(z) = \frac{2}{\pi}\int\limits_0^\infty \frac{u\sin^2 u}{\left(u^2 - z^2\sin^2 u\right)^{3/2}}\, du\,.$$

Relation with C and D:

$$G_{2s} = 2\left(\frac{1-z^2}{z^2} C_{2s-1} + D_{2s-1}\right)$$

Relation D via C

$$D_\lambda(z) = \frac{1-z^2}{z^2}\, C_\lambda(z) + \frac{2}{z^2}\int\limits_0^z x\, C_\lambda(x)\, dx\,.$$

Relation C via D

$$C_\lambda(z) = \frac{z^2}{1-z^2}\,\tilde{C}_\lambda(z) - 2\int\limits_0^z \frac{x^3}{(1-x^2)^2}\,\tilde{C}_\lambda(x)\, dx$$

Inhomogeneous ODE:

$$(1-z^2)\, C'_{2s-1}(z) - z\, C_{2s-1}(z) = \frac{1}{4}\left[z^2 G_{2s}(z)\right]'$$

Solution of C via G:

$$C_{2s-1}(z) = \frac{z^2 G_{2s}(z)}{4(1-z^2)} - \frac{1}{4\sqrt{1-z^2}}\int\limits_0^z \frac{x^3 dx}{(1-x^2)^{3/2}}\, G_{2s}(x)$$

C_1 and D_1

Introducing $A = u^{-2} \sin^2 u$, $Z = z^2$

$$C_1(z) = -\frac{1}{2\pi} \int_0^\infty \frac{du\, A}{(1-A)^2} \left[\frac{2 - ZA(1+A)}{(1-ZA)^{3/2}} - \frac{2}{\sqrt{1-Z}} \right]$$

Another repres.:

$$C_1(z) = -\frac{3z^2}{2\pi} \int_0^\infty du \frac{u(u\cos u - \sin u)\sin u}{(u^2 - z^2 \sin^2 u)^{5/2}} .$$

$$C_1(z) = \frac{1}{4\pi} \frac{z^2}{1-z^2} \int_0^\infty \frac{2u - \sin 2u}{(u^2 - z^2 \sin^2 u)^{3/2}} du$$

Corresponding solution for D_1:

$$D_1(z) = \frac{1}{4\pi} \int_0^\infty \frac{-2u\cos 2u + \sin 2u}{(u^2 - z^2 \sin^2 u)^{3/2}} du$$

Matched asymptotic expansion

$$C_1(z) = \frac{1}{4\pi}\frac{1-\eta^2}{\eta^2}\left[\int_0^{\alpha} + \int_{\alpha}^{\infty}\right]\frac{2u - \sin 2u}{\left(u^2 - z^2\sin^2 u\right)^{3/2}}\,du$$

$\alpha \to 0^+$ and $\alpha/\eta \to \infty$ as $\eta \to 0^+$. In fact, $\alpha = \sqrt{\eta}$

For L_1:

$$L_1 := \int_0^{\alpha} \dots\, du = \eta \int_0^{\alpha/\eta} \frac{2\eta s - \sin 2\eta s}{\left(\eta^2 s^2 - (1-\eta^2)\sin^2 \eta s\right)^{3/2}}\,ds$$

Typical integrals:

$$\int_0^{\alpha/\eta} \frac{s^{2p}}{(1+s^2/3)^{q/2}} ds = \int_0^{\eta^{-1/2}} \dots ds \to \int_0^{\infty} \dots ds$$

For L_2:

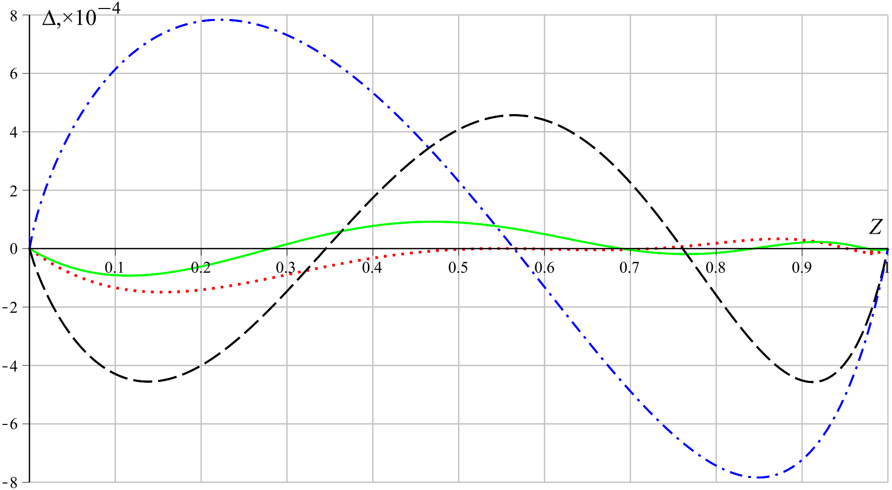
$$L_2 = \sum_{k=0}^{\infty} \frac{(2k+1)!!}{(-1)^k 2^k k!} \eta^{2k} \int_{\alpha}^{\infty} \frac{(2u - \sin 2u)\, du}{(u^2 - \sin^2 u)^{k+3/2}} \sin^{2k} u$$

Final:

$$C_1 = \frac{\sqrt{3}}{\pi}\left[\frac{\eta^{-4}}{3} - \frac{\eta^{-2}}{5} - \frac{1}{525}\eta^{-1} + \right.$$
$$\left. + \frac{1}{35}\ln\eta + \left(\frac{1321}{2100} - \frac{\sqrt{3}}{8}B_1 + \frac{1}{70}\ln\frac{3}{4}\right) + \mathcal{O}(\eta)\right]$$

$$D_1 = \frac{\sqrt{3}}{\pi}\left[\frac{2}{3}\eta^{-2} + \frac{2}{5}\ln\eta + \left(\frac{\tilde{B}_0}{\sqrt{3}} - \frac{14}{15} + \frac{1}{5}\ln\frac{3}{4}\right) - \frac{4}{525}\eta + \right.$$
$$\left. + \left(\frac{491}{450} + \frac{\sqrt{3}}{8}(4\tilde{B}_1 - B_1) + \frac{1}{5}\ln\frac{3}{4}\right)\eta^2 + \mathcal{O}(\eta^3)\right]$$

Approximations



Relative errors for four approximations of C_1 versus $Z = z^2$. The best one (green) has maximal relative error $9.241 \cdot 10^{-5}$

Conclusions

- The general scheme for the integral representations C_{2s+1} and D_{2s+1}
- Concrete integral representations C_1 and D_1
- Asymptotic regimes
- Interpolations of C_1 (rel.10^{-4}) and D_1 (rel.10^{-3})
- Recurrence relations on C_{2s+1} and D_{2s+1}
- Lorentz-Dirac force for the synchrotron motion in three dimensions