Casimir interaction of finite-width cosmic strings

Yuri V. Grats, Pavel Spirin

Moscow State U.

21st Lomonosov Conference
August 24-30, 2023, Moscow, RF

- *Y.V. Grats, P.Spirin*, Phys. Rev. D **108**, (2023) no.4, 045001
- *Y.V. Grats, P.Spirin*, Moscow Univ. Bull. **78**, (2023) no.5, 2350101

Content:

Metric (cylindric coords):

$$ds^2 = dt^2 - dz^2 - d\varrho^2 - \beta^2 \varrho^2 d\varphi^2\,, \qquad 0 < \beta \leqslant 1$$

Geometry:

$$R = 2(1-\beta)\delta_+(\varrho)/\varrho, \qquad \delta\varphi = 2\pi(1-\beta)$$

Phase transition energy scale:

$$\eta^2 = \frac{1-\beta^2}{8\pi G}$$

For $\eta = \eta_{\rm GUT} \sim 10^{16}$ GeV

$$1-\beta \sim 10^{-5} \qquad a \sim 10^{-29}\text{cm}$$

Complement:

$$\beta' \equiv 1-\beta = \frac{\delta\varphi}{2\pi} \qquad \beta' = 4G\mu$$

Single cosmic string's geometry

Conformally Euclidean ($z = \mathrm{const}$) coords: $\varrho \to r$

$$\beta\varrho = R_0(r/R_0)^\beta \,,$$

Metric (conformal coords):

$$ds^2 = dt^2 - dz^2 - \mathrm{e}^{-2(1-\beta)\ln(r/R_0)}(dx^2 + dy^2) \,,$$

where $r^2 = x^2 + y^2$.

Any 2-dimensional surface $\times \mathbb{M}_{1,1}$

$$ds^2 = dt^2 - dz^2 - e^{-\sigma(\boldsymbol{x})}\left(dx^2 + dy^2\right)$$

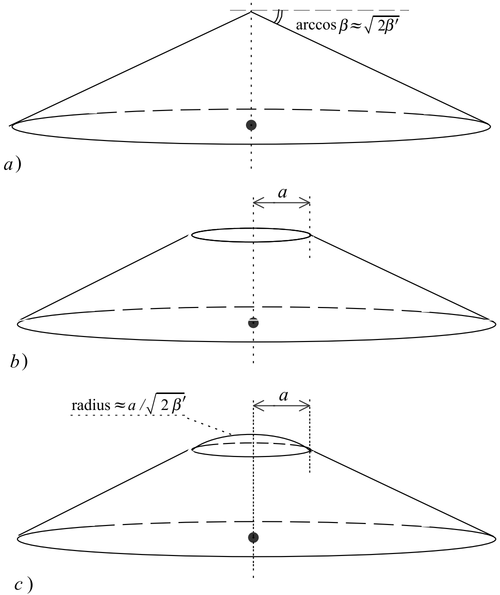
or, equivalently

$$ds^2 = dt^2 - dz^2 - P^2(r)dr^2 + \beta^2 r^2 d\varphi^2 \qquad 0 < \beta \leqslant 1$$

Ricci-scalar:

$$R = e^\sigma \Delta\sigma \simeq \Delta\sigma$$

Finite-width models



Infinitely-thin string:

$$\sigma(r) = 2\beta' \ln r$$
$$P(r) = 1$$
$$R \simeq 4\pi\beta'\,\delta^2(\boldsymbol{x})$$

"Flower-pot" model:

$$P(r) = \beta\theta(a-r) + 1\cdot\theta(r-a)$$
$$R \simeq (2\beta'/a)\,\delta(r-a)$$

"Ballpoint-pen" model:

$$P = \frac{\beta\theta(a-r)}{\sqrt{1-(1-\beta^2)\frac{r^2}{a^2}}} + 1\cdot\theta(r-a)$$
$$\sigma(r) = \beta'\left(1-\frac{r^2}{a^2}\right)\theta(a-r) + +2\beta'\ln\frac{r}{a}\theta(r-a)$$
$$R \simeq (4\beta'/a^2)\theta(a-r)$$

Gauß – Bonnet theorem:

$$\int R\sqrt{g}\, d^2x = 4\pi\beta'$$

Gravitational sterility of cosmic strings:

$$\sigma(\boldsymbol{x}) = \sum_a \sigma_a(|\boldsymbol{x} - \boldsymbol{x}_a|)\,, \qquad \sum_a \beta'_a < 1/8$$

Scalar curvature:

$$R = \mathrm{e}^{\sigma} \sum_a \Delta\sigma_a \simeq \sum_a R_a$$

Conformal factor:

$$\sigma_a(\boldsymbol{x}) = \begin{cases} 2\beta'_a f_a\big(|\boldsymbol{x} - \boldsymbol{x}_a|\big)\,, & |\boldsymbol{x} - \boldsymbol{x}_a| \leqslant a_a\,; \\ 2\beta'_a \ln \dfrac{|\boldsymbol{x} - \boldsymbol{x}_a|}{a_a}\,, & |\boldsymbol{x} - \boldsymbol{x}_a| \geqslant a_a\,, \end{cases}$$

Action for the real scalar field

$$S_\phi = -\frac{1}{2}\int d^d x\,\phi(x)\,L(x,\partial)\,\phi(x)$$

For the massless field: $L(x,\partial) = \sqrt{-g}\,\Box$

The effective action

$$W_{\text{eff}} = -T\mathcal{E}_{\text{vac}}\,,$$

At the other hand,

$$W_{\text{eff}} = \frac{i}{2}\,\mathrm{Sp}\ln L = \frac{i}{2}\,\ln\det L$$

The vacuum energy

$$\mathcal{E}_{\text{vac}} = -\frac{i}{2T}\,\ln\det L\,.$$

Perturbation operator: $L(x,\partial) = \partial^2 + \delta L(x,\partial)$

In our problem

$$\delta L(x,\partial) = \Lambda(\boldsymbol{x})\left(\partial_t^2 - \partial_z^2\right)$$

where

$$\Lambda(\boldsymbol{x}) = \mathrm{e}^{-\sigma(\boldsymbol{x})} - 1 \simeq -\sigma(\boldsymbol{x})$$

The trace:

$$\ln\det L = \mathrm{Sp}\ln(\partial^2) + \mathrm{Sp}\left(\partial^{-2}\,\delta L\right) - \\ - \frac{1}{2}\,\mathrm{Sp}\left(\partial^{-2}\delta L\,\partial^{-2}\delta L\right) + \ldots$$

In Fourier basis

$$\mathcal{E}_{\rm vac} = \frac{i}{4T}\int \frac{d^4k}{(2\pi)^4}\,\frac{d^4p}{(2\pi)^4}\,\frac{(p_0^2 - p_z^2)^2}{p^2\,(p+k)^2}\,\Lambda(k)\,\Lambda(-k)$$

Fourier space

where $\Lambda(k)$ comes from $\delta L(k, \partial) = \Lambda(k)\left(\partial_0^2 - \partial_z^2\right)$:

$$\Lambda(k) = 4\pi^2 \delta(k^0)\, \delta(k^z)\, \Lambda(\boldsymbol{k})$$

2nd delta-function:

$$\delta(k^0)\Big|_{k^0=0} = \frac{1}{2\pi}\int \mathrm{e}^{ik^0 t}\, dt\Big|_{k^0=0} = \frac{1}{2\pi}\int dt = \frac{T}{2\pi}$$

Dimensional regularisation:

$$\mathcal{E}_{\mathrm{vac}}^{\mathrm{reg}} = -\frac{Z\Gamma\left(\frac{4-D}{2}\right)}{120(4\pi)^2}\int \frac{d\boldsymbol{k}}{(2\pi)^2}\left(\boldsymbol{k}^2\right)^{D/2}\sigma(\boldsymbol{k})\,\sigma(-\boldsymbol{k})$$

Two Fourier-integrals encountered:

$$I_{1,2} := \int \frac{d\boldsymbol{k}}{(2\pi)^2}\,|\boldsymbol{k}|^4\,\sigma(\boldsymbol{k})\,\sigma(-\boldsymbol{k}) \times \left\{\begin{array}{l} 1 \\ \ln|\boldsymbol{k}| \end{array}\right\}$$

Back to the coordinate representation

The pole contribution:

$$I_1 = \int d\boldsymbol{x}\,\big[\Delta\sigma(\boldsymbol{x})\big]^2 \simeq \int d\boldsymbol{x}\,\mathrm{R}^2(\boldsymbol{x})$$

The log-contribution:

$$I_2 = -\frac{1}{2\pi}\int d\boldsymbol{x}\,d\boldsymbol{x}'\,\frac{\Delta\sigma\,\Delta'\sigma}{|\boldsymbol{x}-\boldsymbol{x}'|^2} \simeq -\frac{1}{2\pi}\int d\boldsymbol{x}\,d\boldsymbol{x}'\,\frac{\mathrm{R}(\boldsymbol{x})\,\mathrm{R}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^2}$$

Finally:

$$\mathcal{E}^{\mathrm{ren}}_{\mathrm{vac}} = -\frac{Z}{30\,(4\pi)^3}\int d\boldsymbol{x}\,d\boldsymbol{x}'\,\frac{\mathrm{R}(\boldsymbol{x})\,\mathrm{R}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^2}$$

The Casimir energy:

$$\mathcal{E}^{\mathrm{ren}}_{\mathrm{cas}} = -\frac{Z}{15\,(4\pi)^3}\sum_{a<b}\int d\boldsymbol{x}\,d\boldsymbol{x}'\,\frac{\mathrm{R}_a(\boldsymbol{x})\,\mathrm{R}_b(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^2}$$

For two strings of equal radii ($Z = 1$): $\xi := a/d$

$$\mathcal{E}_{\text{cas}} = -\frac{4}{15\pi}\,\frac{\mu_1\mu_2}{d^2}\,F(\xi)$$

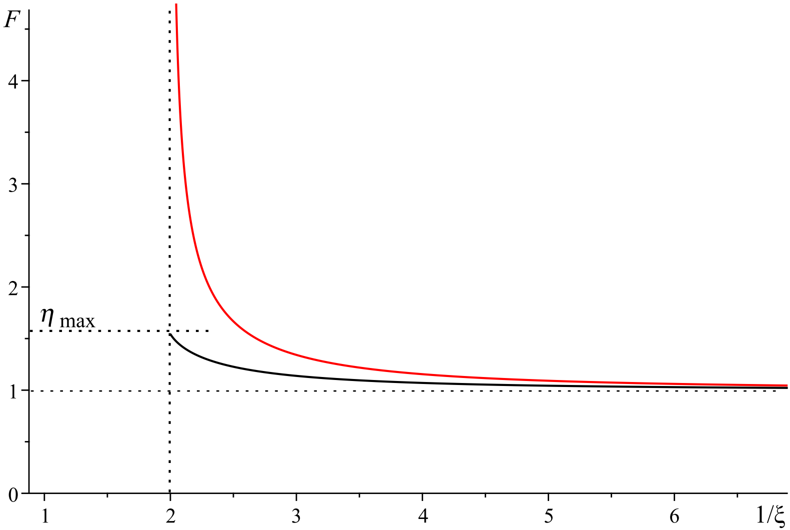
For Flower-pot model:

$$F = \frac{1}{\sqrt{1-4\xi^2}}$$

For Ballpoint-pen model:

$$F = \frac{1}{\xi^2}\left[1 - 2\ln\frac{1+\sqrt{1-4\xi^2}}{2} - \frac{1-\sqrt{1-4\xi^2}}{2\xi^2}\right]$$

Plot of F versus $1/\xi$



red – flower-pot; black – ballpoint-pen

$$\eta_{\max} = 4(2\ln 2 - 1) \approx 1.545\,.$$

Conclusions

- Within the trace-log formalism we extract the vacuum energy
- To compute it, we use the dimensional regularisation
- To the lowest non-vanishing PT-term the Casimir interaction is pairwise
- Terms, proportional to the lowest Schwinger-deWitt terms, are to be neglected
- The supports of the string's curvatures interact with each other what is responsible for the Casimir energy
- In the vicinity of string the FP-model predicts much stronger attraction

Acknowledgment for attention

~~Theorists' lives matter!~~

Thank you!