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V.Yu.Shirokova
Moscow State University, Faculty of Physics, Department of Theoretical Physics

Multiloop calculations of beta-function of N=1 supersymmetric theories, regularized by higher derivatives

NSVZ β-function in $\mathcal{N}=1$ supersymmetric theories

In $\mathcal{N}=1$ supersymmetric theories the β-function is related to the anomalous dimension of the matter superfields by the equation

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}, \qquad \text{where}$$

$$\text{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA}.$$

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

NSVZ equation in different regularizations

It is known that in the $\overline{\text{DR}}$ scheme, which is widely used in supersymmetric theories, the NSVZ relation is incorrect because of the scheme dependence of the renormalization group functions in the highest loops.

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **B 112** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett. **B 386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

The renormalization prescription, in which the NSVZ formula is correct in all loops, is realized on regularizations by higher covariant derivatives, proposed by A.A. Slavnov

A.A.Slavnov, Nucl.Phys., **B 31** (1971) 301; Theor.Math.Phys. **13** (1972) 1064.

in supersymmetric variant:

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745; P.West, Nucl.Phys. **B 268** (1986) 113.

In this case, the NSVZ relation is true in all loops for the RGF, defined in terms of bare coupling constants,

K.V.Stepanyantz, Nucl.Phys. **B 852** (2011) 71; Nucl. Phys. **B 909** (2016), 316-335.
K.V.Stepanyantz, Eur. Phys. J. **C 80** (2020) no.10, 911; JHEP **2001** (2020) 192.

and also in terms of renormalized coupling constants when using the HD+MSL scheme.

A.L.Kataev and K.V.Stepanyantz, Nucl. Phys. B **875** (2013) 459.

$\mathcal{N}=1$ SQED with N_f flavors

The simplest particular case of the $\mathcal{N}=1$ gauge theory is the $\mathcal{N}=1$ supersymmetric electrodynamics (SQED) with N_f flavors, which (in the massless case) is described by the action

$$S=\frac{1}{4e_0^2}\text{Re}\int d^4x\,d^2\theta\,W^aW_a+\sum_{f=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\left(\phi_f^*e^{2V}\phi_f+\widetilde{\phi}_f^*e^{-2V}\widetilde{\phi}_f\right),$$

where V is a real gauge superfield, ϕ_f and $\widetilde{\phi}_f$ with $f=1,\ldots,N_f$ are chiral matter superfields with opposite U(1) charges, and $W_a=\bar{D}^2D_aV/4$. In our notation the bare and renormalized coupling constants are denoted by e_0 and e, respectively.

Regularization and gauge fixing

In order to regularize the theory by higher derivatives, it is necessary to add the higher derivative term to the action:

$$S_{\text{reg}} = \frac{1}{4e_0^2}\text{Re}\int d^4x\, d^2\theta\, W^a R(\partial^2/\Lambda^2) W_a + \sum_{f=1}^{N_f} \frac{1}{4}\int d^4x\, d^4\theta \left(\phi_f^* e^{2V}\phi_f + \widetilde{\phi}_f^* e^{-2V}\widetilde{\phi}_f\right),$$

where $R(\partial^2/\Lambda^2)$ is a regulator, e.g. $R = 1 + \partial^{2n}/\Lambda^{2n}$.
Another similar regulator function appears in the gauge fixing term

$$S_{\text{gf}} = -\frac{1}{32\xi_0 e_0^2}\int d^4x\, d^4\theta\, D^2 V K(\partial^2/\Lambda^2)\bar{D}^2 V,$$

where ξ_0 is the bare gauge parameter. The minimal (Feynman) gauge corresponds to $\xi_0 = 1$ and $R(x) = K(x)$. However, we will make calculations for an arbitrary ξ_0 and $K(x) \neq R(x)$.

Pauli–Villars determinants

Adding the higher derivative term allows to remove all divergences beyond the one-loop approximation. To remove one-loop divergences, we insert in the generating functional the Pauli–Villars determinants:

$$Z[\text{sources}] = \int DV\, \Big(\prod_{\alpha=1}^{N_f} D\phi_\alpha D\widetilde{\phi}_\alpha\Big)\, \text{Det}(PV,M)^{N_f} \exp\Big(iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{ист}}\Big)$$

$$\text{Det}(PV,M)^{-1} = \int D\Phi\, D\widetilde{\Phi}\, \exp(iS_\Phi).$$

Here the action for the massive Pauli–Villars superfields is given be the expression

$$S_\Phi = \frac{1}{4}\int d^4x\, d^4\theta\, \Big(\Phi^* e^{2V}\Phi + \widetilde{\Phi}^* e^{-2V}\widetilde{\Phi}\Big) + \Big(\frac{M}{2}\int d^4x\, d^2\theta\, \widetilde{\Phi}\,\Phi + \text{c.c.}\Big),$$

and it is important that the ratio of the Pauli–Villars mass M to the regularization parameter Λ should not depend on the coupling constant.

Renormalization

The considered theory is renormalizable.

A. A. Slavnov, Nucl. Phys. **B 97** (1975) 155.

Therefore the ultraviolet divergences can be absorbed into the renormalization of the coupling constant, of the gauge parameter, and of the chiral matter superfields ϕ_α and $\widetilde{\phi}_\alpha$. All chiral superfields have the same renormalization constant Z, such that $\phi_{\alpha,R} = \sqrt{Z}\phi_\alpha$, $\widetilde{\phi}_{\alpha,R} = \sqrt{Z}\widetilde{\phi}_\alpha$ for all values of $\alpha = 1, \ldots, N_f$.

The NSVZ relation in $\mathcal{N}=1$ SQED with N_f flavours

It is convenient to encode the ultraviolet divergences in RGFs. It is neccesary to distinguish between RGFs defined in terms of the bare coupling constant,

$$\beta(\alpha_0) = \frac{d\alpha_0}{d\ln\Lambda}\bigg|_{\alpha=\text{const}}; \qquad \gamma(\alpha_0) = -\frac{d\ln Z}{d\ln\Lambda}\bigg|_{\alpha=\text{const}},$$

and the ones standardly defined in terms of the renormalized coupling constant by the equations

$$\widetilde{\beta}(\alpha) = \frac{d\alpha}{d\ln\mu}\bigg|_{\alpha_0=\text{const}}; \qquad \widetilde{\gamma}(\alpha) = \frac{d\ln Z}{d\ln\mu}\bigg|_{\alpha_0=\text{const}},$$

where μ is a renormalization point.

A.L.Kataev and K.V.Stepanyantz, Nucl. Phys. B **875** (2013) 459.

In considered theory and in the case of using the higher derivative regularization described above they satisfy the NSVZ equation

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi}\Big(1-\gamma(\alpha_0)\Big)$$

M.A.Shifman, A.I.Vainshtein, V.I.Zakharov, JETP Lett. **42** (1985) 224; Phys.Lett. B **166** (1986) 334.

Method of calculation of β-function based on vacuum supergraph calculation

There is very simple method for calculation of β-function.

K. V. Stepanyantz, JHEP **1910** (2019) 011.

This method is based on calculating supergraphs with no external lines and than to act on them by specially constructed operator. In the abelian case we follow such steps:

1. First, we calculate a supergraph with an insertion of $\theta^4 v^2$, where v is function which slowly decrease at a very large scale $R \to \infty$ and $\theta^4 \equiv \theta^a \theta_a \bar{\theta}^{\dot{b}} \bar{\theta}_{\dot{b}}$.
2. Then we apply operator:

$$\sum_{i=1}^{M} \frac{\partial^2}{\partial Q_{\mu i}^2},$$

where the index i numerates the matter loops, M — total number of matter loops.

3. Finally the contribution to the function $\beta(\alpha_0)/\alpha_0^2$ corresponding to the considered supergraph is obtained by differentiating the result with respect to $\ln \Lambda$ and multiplying it to the factor $-2\pi/\mathcal{V}_4$, where

$$\mathcal{V}_4 \equiv \int d^4x\, v^2$$

S.S.Aleshin, *et al.* Nucl. Phys. B **956** (2020), 115020

Method of calculation of β-function based on vacuum supergraph calculation

Only singular near zero contributions give non-trivial results. This is because after using the formula

$$\frac{\partial^2}{\partial Q_\mu^2}\frac{1}{Q^2} = -4\pi^2\delta^4(Q)$$

C. M. Bender, R. W. Keener and R. E. Zippel, Phys. Rev. D **15** (1977), 1572
K. V. Stepanyantz, JHEP **1910** (2019) 011.

one can take one of the loop integrals and all the parts with non-singular contributions will vanish. After taking integral we will have $L-1$ integrals (where L is number of loops).

Computer program for supergarph calculations

We use special computer program written earlier for calculations in the framework of $N = 1$ superspace.

I. E. Shirokov, Program. Comput. Software **49** (2023), 122-130.

Initially the program could calculate two-point Green function of matter fields. Several changes were made in it to make it possible to calculate supergraphs with no external legs, that are needed in mentioned above method.

Four-loop β-function, terms proportional to $(N_f)^2$ and $(N_f)^3$

$$
\begin{aligned}
&\Delta_{(N_f)^2,(N_f)^3}\frac{\beta(\alpha_0)}{\alpha_0^2} = 8\pi(N_f)^2\frac{d}{d\ln\Lambda}\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{e_0^4}{K^2R_K^2}\frac{\partial^2}{\partial Q^\mu\partial Q_\mu}\Big(\frac{1}{Q^2(Q+K)^2}+\text{n.-s. c.}\Big)\\
&\times\Big(\frac{1}{((L+K)^2+M^2)(L^2+M^2)}-\frac{1}{L^2(L+K)^2}\Big)+16\pi(N_f)^2\frac{d}{d\ln\Lambda}\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4U}{(2\pi)^4}\frac{e_0^6}{R_K^2K^2}\\
&\times\frac{1}{R_LL^2}\Big(\frac{\partial^2}{\partial Q^\mu\partial Q_\mu}+\frac{\partial^2}{\partial U^\mu\partial U_\mu}\Big)\Big(\frac{K^2+L^2-2(Q+K+L)^2}{U^2(U+K)^2Q^2(Q+K)^2(Q+L)^2(Q+K+L)^2}-\frac{1}{(Q+K+L)^2}\\
&\times\frac{K^2+L^2-2(Q+K+L)^2}{Q^2(Q+L)^2(U^2+M^2)(Q+K)^2((U+K)^2+M^2)}-\frac{K^2+L^2-2(Q+K+L)^2}{U^2(U+K)^2(Q^2+M^2)((Q+K+L)^2+M^2)}\\
&\times\frac{1}{((Q+L)^2+M^2)((Q+K)^2+M^2)}-\frac{4M^2}{U^2(U+K)^2(Q^2+M^2)^2((Q+K)^2+M^2)((Q+L)^2+M^2)}\\
&+\text{n.-s. c.}\Big)+16\pi(N_f)^3\frac{d}{d\ln\Lambda}\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4U}{(2\pi)^4}\frac{e_0^6}{R_K^3K^2}\frac{\partial^2}{\partial U^\mu\partial U_\mu}\Big(\frac{1}{U^2(U+K)^2}+\text{n.-s. c.}\Big)\\
&\times\Big(\frac{1}{L^2(L+K)^2}-\frac{1}{(L^2+M^2)((L+K)^2+M^2)}\Big)\Big(\frac{1}{Q^2(Q+K)^2}-\frac{1}{((Q+K)^2+M^2)(Q^2+M^2)}\Big)\\
&+O(e_0^8)
\end{aligned}
$$

Four-loop β-function, terms proportional to $(N_f)^1$

$$
\begin{aligned}
&\Delta_{(N_f)^1}\frac{\beta(\alpha_0)}{\alpha_0^2} = 2\pi N_f \frac{d}{d\ln\Lambda}\int\frac{d^4Q}{(2\pi)^4}\frac{\partial^2}{\partial Q^\mu\partial Q_\mu}\frac{\ln(Q^2+M^2)}{Q^2} + 4\pi N_f \frac{d}{d\ln\Lambda}\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{e_0^2}{K^2R_K}\frac{\partial^2}{\partial Q^\mu\partial Q_\mu}\\
&\times\Big(\frac{1}{Q^2(Q+K)^2}+\text{n.-s.c.}\Big) + 8\pi N_f\frac{d}{d\ln\Lambda}\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{e_0^4}{K^2R_KL^2R_L}\frac{\partial^2}{\partial Q^\mu\partial Q_\mu}\Big(\frac{1}{Q^2(Q+K)^2(Q+L)^2}\\
&-\frac{K^2}{Q^2(Q+K)^2(Q+L)^2(Q+K+L)^2}+\text{n.-s.c.}\Big) + 8\pi N_f\frac{d}{d\ln\Lambda}\int\frac{d^4Q}{(2\pi)^4}\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4U}{(2\pi)^4}\frac{e_0^6}{K^2R_KL^2R_LQ^2R_Q}\\
&\times\frac{\partial^2}{\partial U^\mu\partial U_\mu}\Big(\frac{4}{3U^2(U+K)^2(U+L)^2(U+Q)^2}-\frac{3}{U^2(U+K)^2(U+K+L)^2(U+Q)^2}-\frac{4}{U^2(U+K+L)^2(U+Q+K)^2}\\
&\times\frac{1}{(U+L)^2}+\frac{2L^2Q^2}{U^2(U+L)^2(U+K)^2(U+Q)^2(U+K+L)^2(U+Q+L)^2}-\frac{1}{U^2(U+Q)^2(U+L)^2(U+K+L)^2}\\
&-\frac{4L^2}{U^2(U+Q)^2(U+L)^2(U+K)^2(U+K+L)^2}+\frac{(K+Q)^2\times[3(U+L+K)^2-(U+K)^2-(U+L)^2+U^2+L^2]}{U^2(U+L)^2(U+K)^2(U+K+Q+L)^2(U+K+L)^2(U+Q+K)^2}\\
&+\frac{2K^4}{U^2(U+L)^2(U+K)^2(U+Q)^2(U+K+L)^2(U+Q+K)^2}+\frac{2(Q+K+L)^4}{3U^2(U+L)^2(U+K)^2(U+K+Q+L)^2(U+Q+L)^2}\\
&\times\frac{1}{(U+Q+K)^2}+\frac{16[(U+K)^2+(U+L)^2+(U+Q)^2-U^2-L^2-K^2-Q^2]}{3U^2(U+L)^2(U+K)^2(U+Q+L)^2(U+Q+K)^2}+\text{n.-s.c.}\Big)+O(e_0^8)
\end{aligned}
$$

Lower order contributions were obtained earlier

S.S.Aleshin, *et al.* Nucl. Phys. B **956** (2020), 115020

But we recalculated them independently using program.

Four-loop β-function

After calculating one of the loop integrals using differential operator the result takes form

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi} - \frac{2N_f}{\pi}\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{e_0^2}{K^4R_K} - \frac{2N_f}{\pi}\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{e_0^4}{R_KR_L}\left(\frac{2}{K^2L^4(K+L)^2}\right.$$

$$\left.-\frac{1}{K^4L^4}\right) - \frac{4(N_f)^2}{\pi}\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{e_0^4}{R_K^2K^4}\left(\frac{1}{L^2(L+K)^2} - \frac{1}{(L^2+M^2)((L+K)^2+M^2)}\right)$$

$$-\frac{8N_f}{\pi}\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\frac{e_0^6}{R_KR_LR_Q}\left[-\frac{1}{3K^4L^4Q^4} + \frac{1}{K^4L^2Q^4(Q+L)^2} + \frac{1}{K^2L^4(K+L)^2}\right.$$

$$\left.\times\frac{1}{(Q+K+L)^2}\left(\frac{1}{Q^2} - \frac{2}{(Q+L)^2}\right)\right] - \frac{16(N_f)^2}{\pi}\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\frac{e_0^6\,K_\mu L^\mu}{R_K^2R_LK^4L^4(K+L)^2}$$

$$\times\left(\frac{1}{Q^2(Q+K)^2} - \frac{1}{(Q^2+M^2)((Q+K)^2+M^2)}\right) - \frac{8(N_f)^2}{\pi}\frac{d}{d\ln\Lambda}\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\frac{e_0^6}{R_K^2R_LK^4}$$

$$\times\frac{1}{L^2}\left(\frac{2(Q+K+L)^2-K^2-L^2}{Q^2(Q+K)^2(Q+L)^2(Q+K+L)^2} - \frac{2(Q+K+L)^2-K^2-L^2}{(Q^2+M^2)((Q+K)^2+M^2)((Q+L)^2+M^2)}\right.$$

$$\left.\times\frac{1}{((Q+K+L)^2+M^2)} + \frac{4M^2}{(Q^2+M^2)^2((Q+K)^2+M^2)((Q+L)^2+M^2)}\right) + \frac{8(N_f)^3}{\pi}\frac{d}{d\ln\Lambda}$$

$$\times\int\frac{d^4K}{(2\pi)^4}\frac{d^4L}{(2\pi)^4}\frac{d^4Q}{(2\pi)^4}\frac{e_0^6}{R_K^3K^4}\left(\frac{1}{Q^2(Q+K)^2} - \frac{1}{(Q^2+M^2)((Q+K)^2+M^2)}\right)\left(\frac{1}{L^2(L+K)^2}\right.$$

$$\left.-\frac{1}{(L^2+M^2)((L+K)^2+M^2)}\right) + O(e_0^8).$$

Four-loop β-function and NSVZ-relation

All the remaining integrals were calculated in

I. E. Shirokov and K. V. Stepanyantz, JHEP 2204 (2022) 108.

using the Chebyshev polynomials method.

J. L. Rosner, Annals Phys. **44** (1967), 11.

$$\beta(\alpha_0) = \frac{\alpha_0^2 N_f}{\pi} + \frac{\alpha_0^3 N_f}{\pi^2} - \frac{\alpha_0^4 N_f}{2\pi^3} - \frac{\alpha_0^4 (N_f)^2}{\pi^3}\Big(\ln a + 1 + \frac{A_1}{2}\Big) + \frac{\alpha_0^5 N_f}{2\pi^4} + \frac{\alpha_0^5 (N_f)^2}{\pi^4}$$
$$\times\Big(\ln a + \frac{3}{4} + C\Big) + \frac{\alpha_0^5 (N_f)^3}{\pi^4}\Big((\ln a + 1)^2 - \frac{A_2}{4} + D_1 \ln a + D_2\Big) + O(\alpha_0^6).$$

where A_1, A_2, C, D_1, and D_2 are numerical parameters depending on the regulator function $R(x)$.

$$\gamma(\alpha_0) = -\frac{\alpha_0}{\pi} + \frac{\alpha_0^2}{2\pi^2} + \frac{\alpha_0^2 N_f}{\pi^2}\Big(\ln a + 1 + \frac{A_1}{2}\Big) - \frac{\alpha_0^3}{2\pi^3} - \frac{\alpha_0^3 N_f}{\pi^3}\Big(\ln a + \frac{3}{4} + C\Big)$$
$$-\frac{\alpha_0^3 (N_f)^2}{\pi^3}\Big((\ln a + 1)^2 - \frac{A_2}{4} + D_1 \ln a + D_2\Big) + O(\alpha_0^4),$$

comparison verify validity of NSVZ relation:

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{N_f}{\pi}\Big(1 - \gamma(\alpha_0)\Big)$$

Conclusion

- Using special computer program, written previously by I.E. Shirokov and redesigned by author, and also using standard rules of inserting differential operator, integrals that contribute to beta function were obtained.
- Four-loop β-function defined in terms of bare couplings was calculated and compared to three-loop anomalous dimension.
- It appears to be the highest loop verification of NSVZ-relation.
- This result can be treated as a verification of this special method of β-function calculation and of used software.

Thank you for the attention!