



# Recent results of charm physics at BESIII

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(On behalf of BESIII Collaboration)

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# Outline

## Introduction

## (Semi-) leptonic decays

□  $D_S^+ \rightarrow \tau^+ \nu_\tau$

•  $\tau^+ \rightarrow e^+ \nu_e \bar{\nu}_\tau$

•  $\tau^+ \rightarrow \pi^+ \pi^0 \bar{\nu}_\tau$

•  $\tau^+ \rightarrow \pi^+ \bar{\nu}_\tau$  }  
}

□  $D_S^+ \rightarrow \mu^+ \nu_\mu$

□  $D_{(s)} \rightarrow Pl^+ \nu_l$  ( $P = \eta, X, K_1(1270)^-$ )

arXiv: 2106.02218 [hep-ex]

arXiv: 2105.07178 [hep-ex], accepted by PRD

arXiv: 2102.11734 [hep-ex], accepted by PRD

PRL124(2020)231801

PRD104(2021)012003

arXiv: 2102.10850 [hep-ex]

## Hadronic decays

□  $D \rightarrow K_{S,L}^0 \pi^+ \pi^-$

□  $D \rightarrow K_{S,L}^0 K^+ K^-$

□  $D \rightarrow K^+ \pi^+ \pi^+ \pi^-$  and  $D \rightarrow K^- \pi^+ \pi^0$

□  $D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0$

PRL124(2020)241802; PRD101(2020)112002

PRD102(2020)052008

JHEP05(2021)164

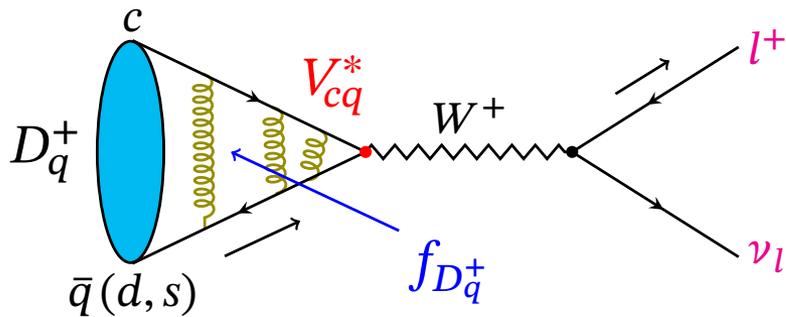
PRL125(2020)141802

## Summary

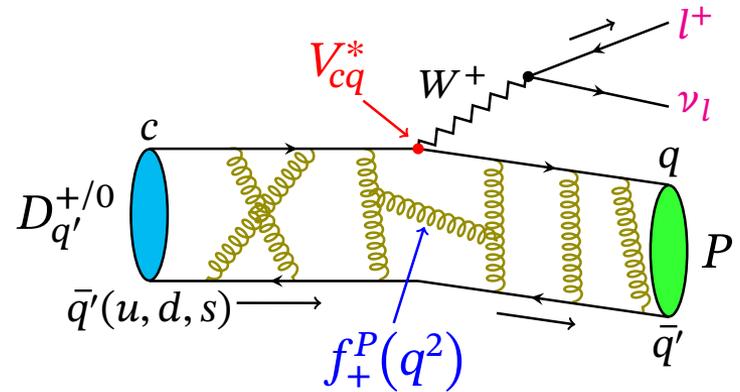
# Main goals

In the SM:

$D_{(s)}$  pure leptonic decay



$D_{(s)}$  semi-leptonic decay



$$\Gamma(D_{(s)}^+ \rightarrow l^+ \nu_l) \propto |f_{D_{(s)}^+}|^2 \cdot |V_{cd(s)}|^2$$

$$\Gamma(D_{(s)} \rightarrow P l^+ \nu_l) \propto |f_+(q^2)|^2 \cdot |V_{cd(s)}|^2$$

- ❖ Decay constant  $f_{D_{(s)}^+}$ , form factor  $f_+(0)$ : calibrate Lattice QCD
- ❖ CKM matrix element  $|V_{cd(s)}|$ : test the unitarity of the CKM matrix
- ❖ Lepton flavor universality (LFU) test.

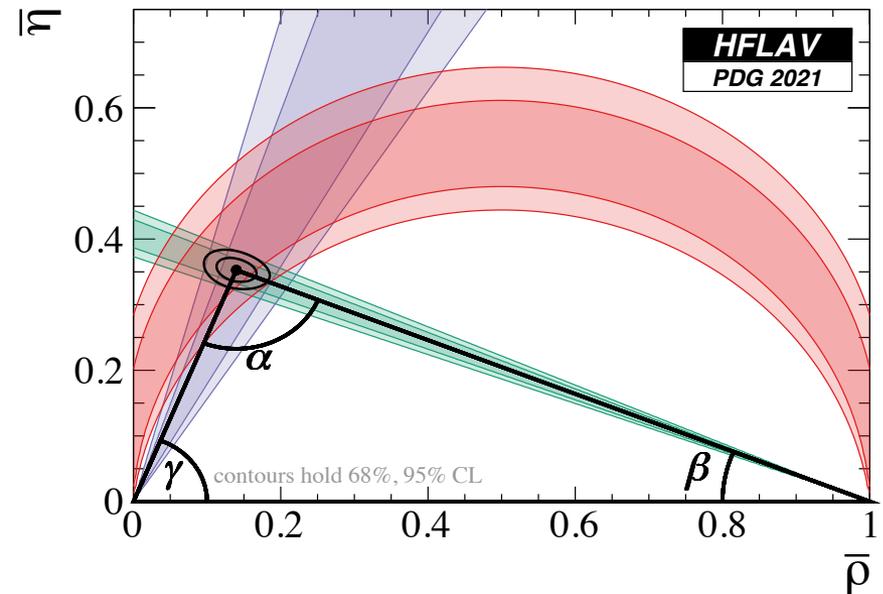
# Hadronic decays of charm mesons

◆ **Strong phase measurement** with quantum correlated  $\psi(3770) \rightarrow D^0 \bar{D}^0$  is crucial in the model-independent determinations of  $\gamma/\phi_3$  and charm mixing/direct CPV.

- In the SM, CP violation is studied by measuring CKM matrix, represented by unitarity triangle in complex plane. The angle is the only one that can be extracted from tree-level processes, for which the contribution of non-SM effect is small.

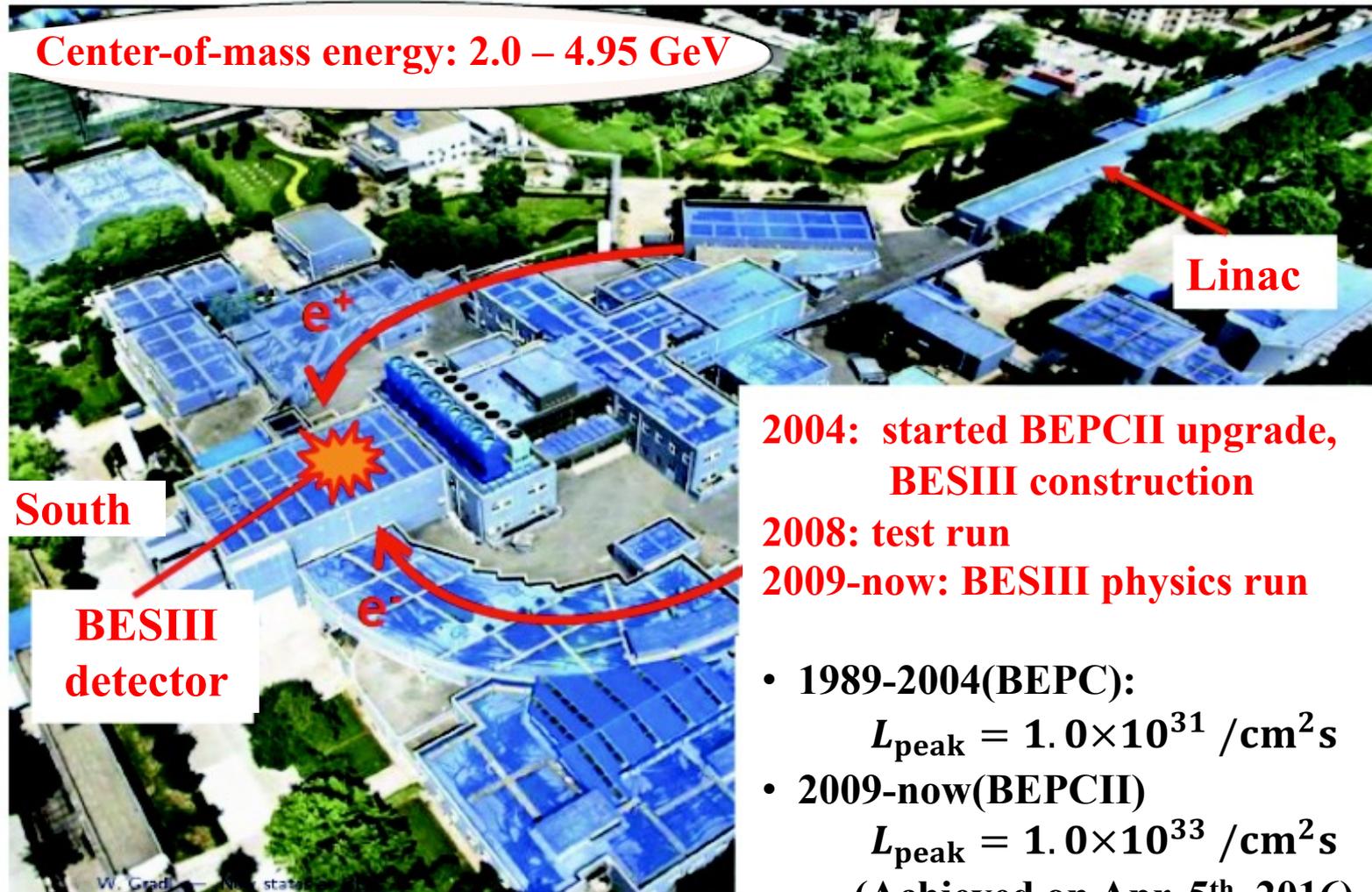
◆ **Probe non-perturbative QCD**

- Help to understand hadron spectra
- Study SU(3) flavor symmetry



# Beijing Electron Positron Collider (BEPCII) in China

A double-ring collider with high luminosity



# BESIII Detector

From inner to outside[1]:

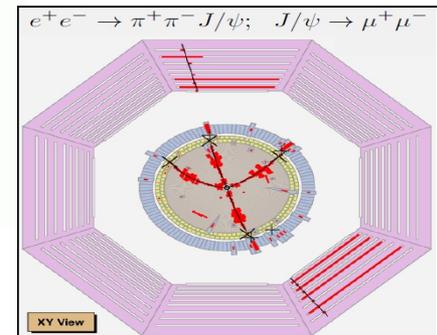
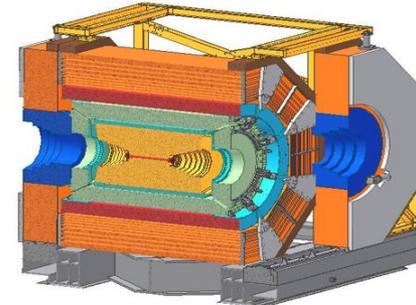
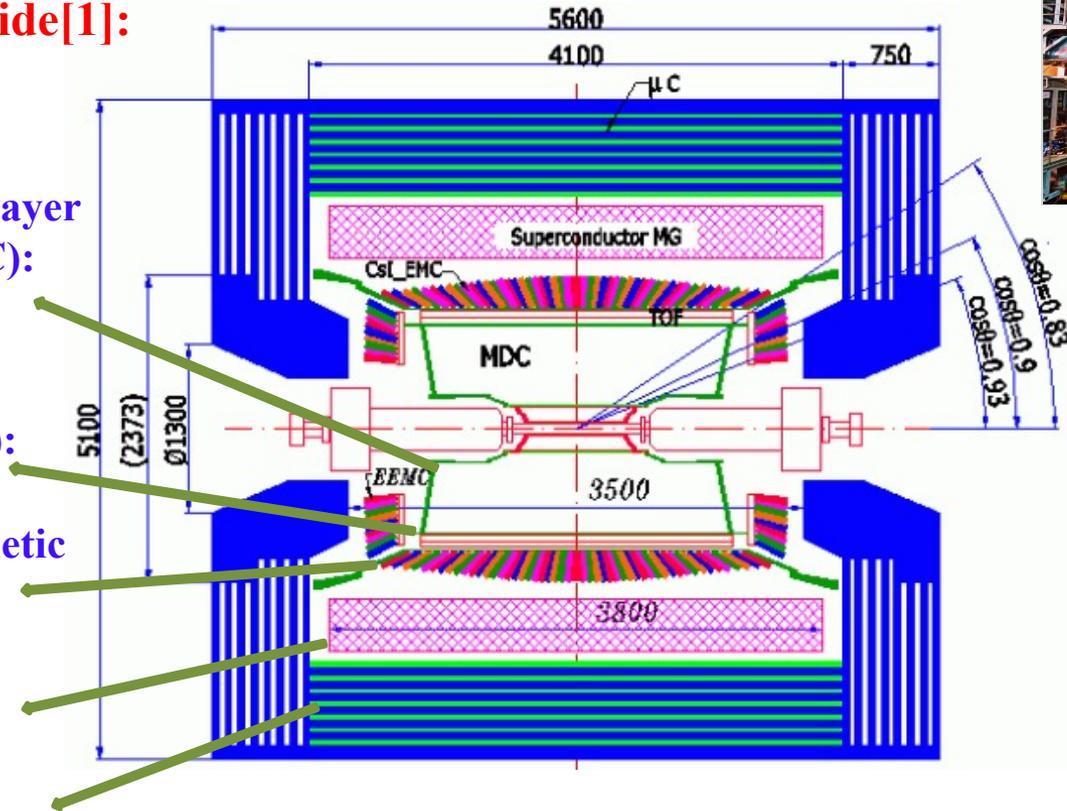
Helium-based multilayer drift chamber (MDC):

Plastic scintillator time-of-flight (TOF):

CsI (Tl) electromagnetic calorimeter (EMC):

Superconducting solenoidal magnet:

Muon Chamber (MUC):



[1] M. Ablikim *et al.* (BESIII Collaboration), Nucl. Instr. Meth. A614, 345 (2010).

# $D^{0(+)}$ and $D_s^+$ data set at BESIII

$\sqrt{s}(\text{GeV})$	Integrated luminosity	Decay chain of interest
3.773	2.93 fb <sup>-1</sup>	$e^+e^- \rightarrow \psi(3770) \rightarrow D^0\bar{D}^0$
		$e^+e^- \rightarrow \psi(3770) \rightarrow D^+D^-$
$\sqrt{s}(\text{GeV})$	Integrated luminosity(pb <sup>-1</sup> )	$e^+e^- \rightarrow D_s^{*\pm}D_s^\mp$ <b>Total: 6.32 fb<sup>-1</sup></b>
4.178	$3189.0 \pm 0.9 \pm 31.9$	
4.189	$526.7 \pm 0.1 \pm 2.2$	
4.199	$526.0 \pm 0.1 \pm 2.1$	
4.209	$517.1 \pm 0.1 \pm 1.8$	
4.219	$514.6 \pm 0.1 \pm 1.8$	
4.226	$1047.3 \pm 0.1 \pm 10.2$	

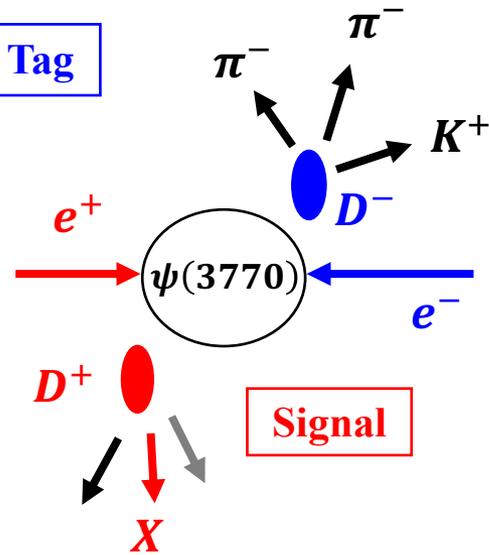
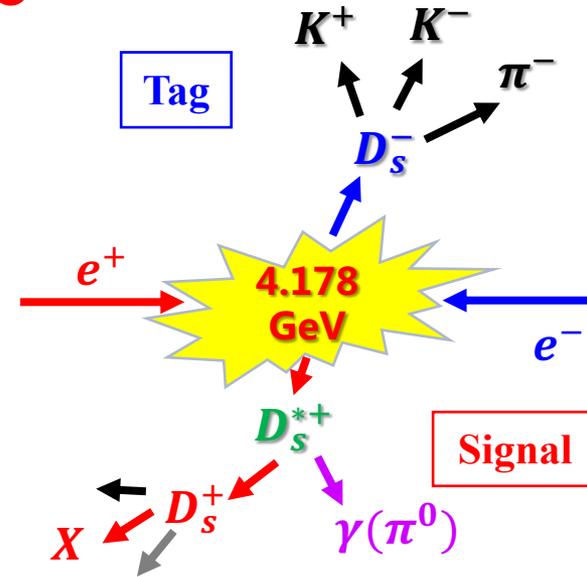
# Analysis technique

Similar for the hadronic decays

Charge conjugated processes are implied

The signal branching fraction:

$$B_{\text{sig}} = \frac{N_{\text{DT}}^{\text{signal}}}{N_{D(s)}^{\text{ST}} \times \epsilon_{\text{sig}}}$$

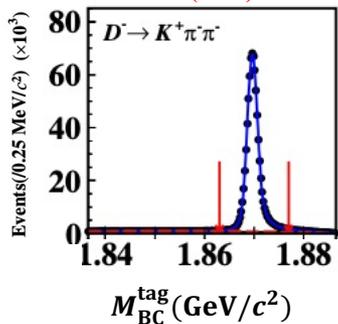


- **Single tag (ST):** fully reconstruct one  $D^-$

$$\Delta E = E_{D^-} - E_{\text{beam}}$$

$$M_{\text{BC}} = \sqrt{E_{\text{beam}}^2 - |\vec{p}_{D^-}|^2}$$

PRL124(2020)231801



- **Double tag (DT):** in the recoil ST  $D_{(s)}^-$ , analyze the signal  $D_{(s)}^+$

$$M^2 = E_{\text{miss}}^2 - |\vec{p}_{\text{miss}}|^2$$

$$E_{\text{miss}} = E_{\text{cm}} - \sqrt{|\vec{p}_{D_{(s)}^-}|^2 + M_{D_{(s)}}^2} - E_X$$

$$\vec{p}_{\text{miss}} = -\vec{p}_{D_{(s)}^-} - \vec{p}_X$$

$$U_{\text{miss}} = E_{\text{miss}} - |\vec{p}_{\text{miss}}|$$

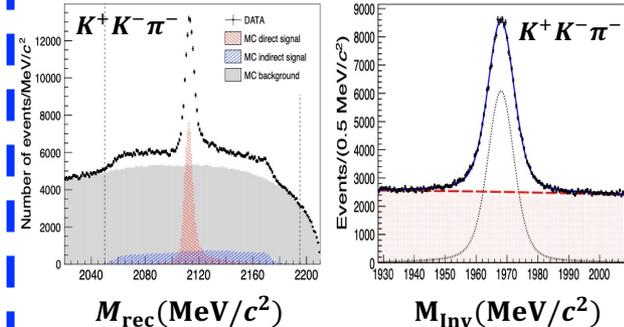
or other variables

- **Single tag (ST):** fully reconstruct one  $D_s^-$

$$M_{\text{rec}} = \sqrt{\left(E_{\text{cm}} - \sqrt{|\vec{p}_{D_s^-}|^2 + m_{D_s^-}^2}\right)^2 - |\vec{p}_{D_s^-}|^2}$$

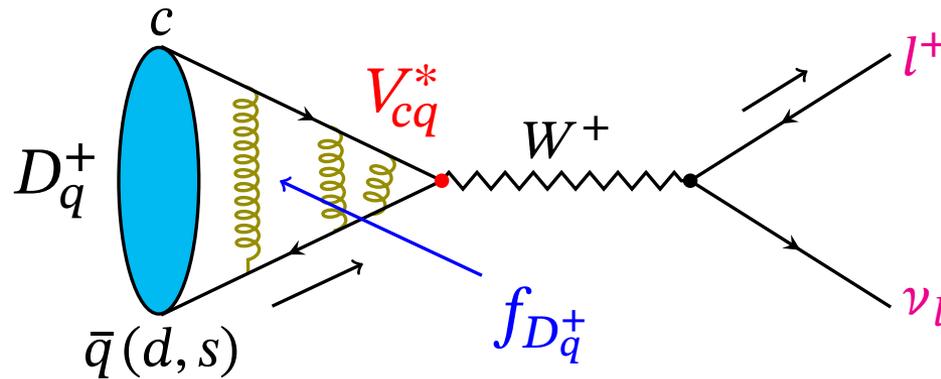
arXiv: 2102.11734 [hep-ex]

PRD104(2021)012003



# Pure leptonic $D_s^+$ decays

In the SM



$$\Gamma(D_{(s)}^+ \rightarrow l^+ \nu_l) = \frac{G_F^2 f_{D_{(s)}^+}^2}{8\pi} |V_{cd(s)}|^2 m_l^2 m_{D_{(s)}^+} \left(1 - \frac{m_l^2}{m_{D_{(s)}^+}^2}\right)^2$$

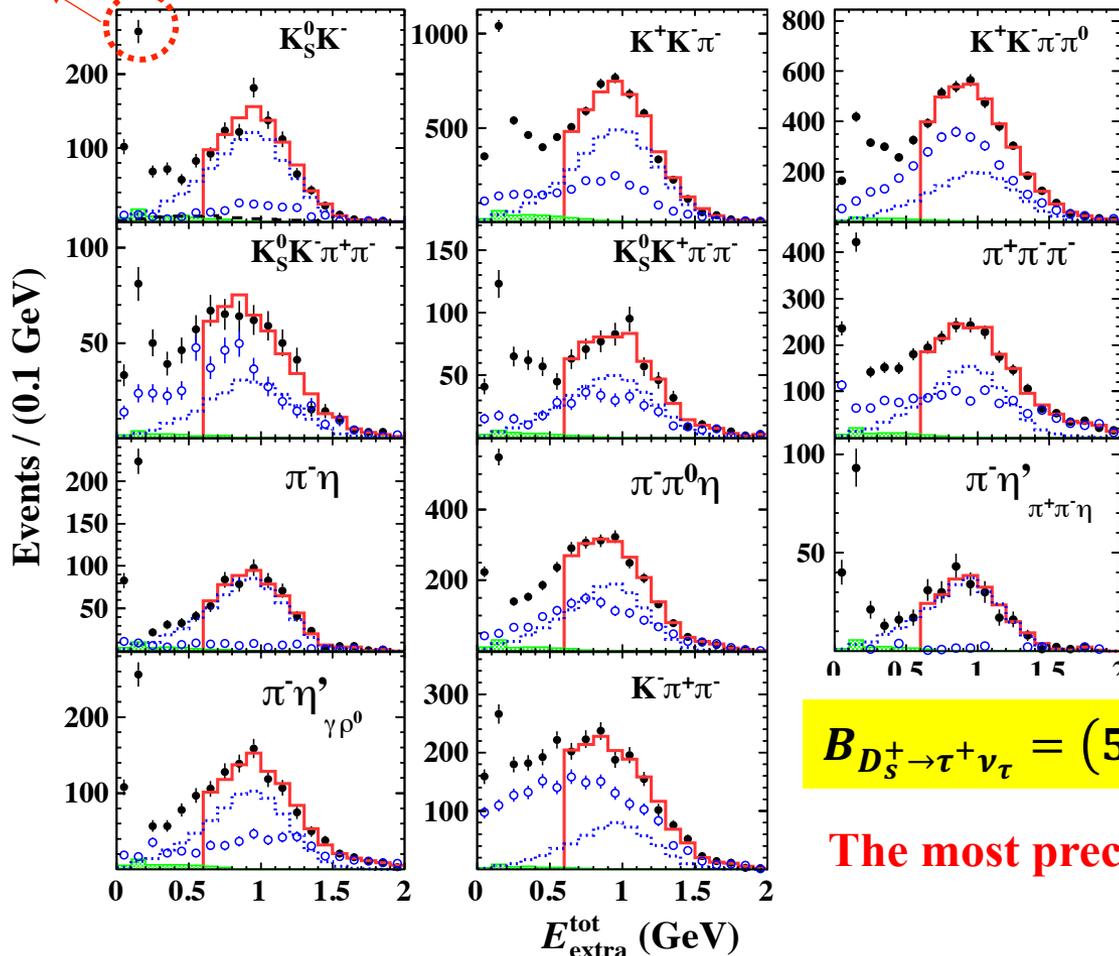
# $D_s^+ \rightarrow \tau^+ \nu_\tau$ via $\tau^+ \rightarrow e^+ \nu_e \bar{\nu}_\tau$

arXiv: 2106.02218 [hep-ex]

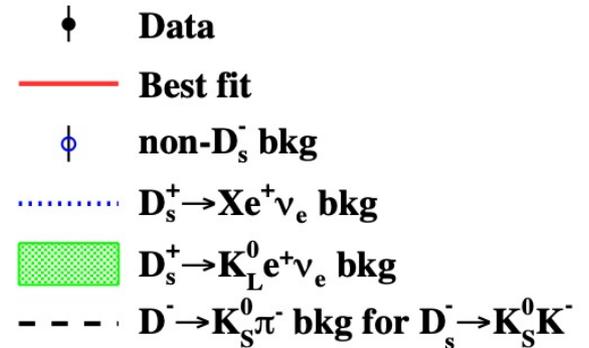
**$E_{\text{extra}}^{\text{tot}}$** : the total energy of the good EMC showers, excluding those associated with the ST  $D_s^-$  candidates and those within  $5^\circ$  of the initial direction of the positron.

**DT yield in signal  $E_{\text{extra}}^{\text{tot}} < 0.4$  GeV**:  $N_{\text{DT}} = N_{\text{DT}}^{\text{tot}} - N_{\text{DT}}^{\text{non-}D_s^-} - N_{\text{DT}}^{K_L^0 e^+ \nu_e} - N_{\text{DT}}^{X e^+ \nu_e}$

Signal peak



The background yield of  $D_s^+ \rightarrow X e^+ \nu_e$  is **extrapolated** from the fit to  $E_{\text{extra}}^{\text{tot}} > 0.6$  GeV.



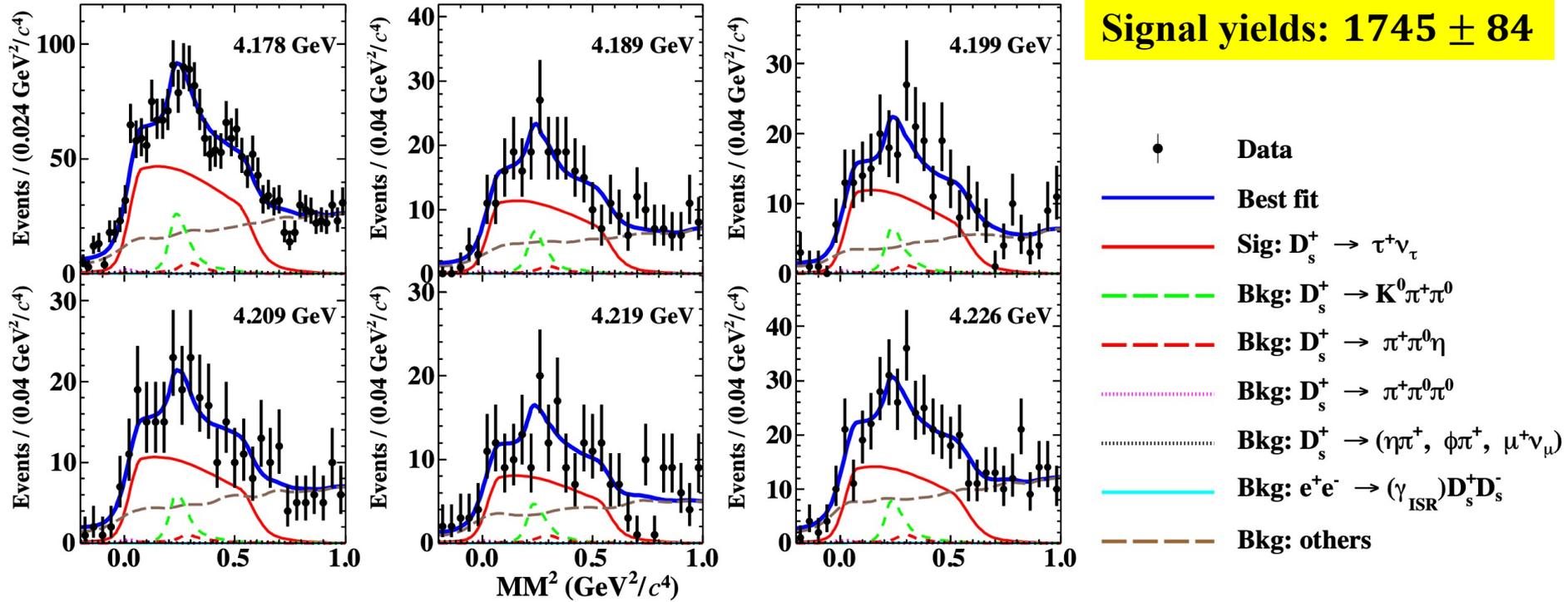
$$B_{D_s^+ \rightarrow \tau^+ \nu_\tau} = (5.27 \pm 0.10_{\text{stat.}} \pm 0.12_{\text{syst.}})\%$$

The most precise result to date

# $D_s^+ \rightarrow \tau^+ \nu_\tau$ via $\tau^+ \rightarrow \pi^+ \pi^0 \bar{\nu}_\tau$

arXiv: 2105.07178 [hep-ex], accepted by PRD

- **Simultaneous** fit to the  $MM^2$  for six energy points shared with a **common** leptonic branching fraction.



$$MM^2 = E_{\text{miss}}^2 - |\vec{p}_{\text{miss}}|^2$$

$$E_{\text{miss}} = E_{\text{cm}} - \sqrt{|\vec{p}_{D_s^-}|^2 + M_{D_s}^2} - E_\gamma - E_{\pi^+ \pi^0}$$

$$\vec{p}_{\text{miss}} = -\vec{p}_{D_s^-} - \vec{p}_\gamma - \vec{p}_{\pi^+ \pi^0}$$

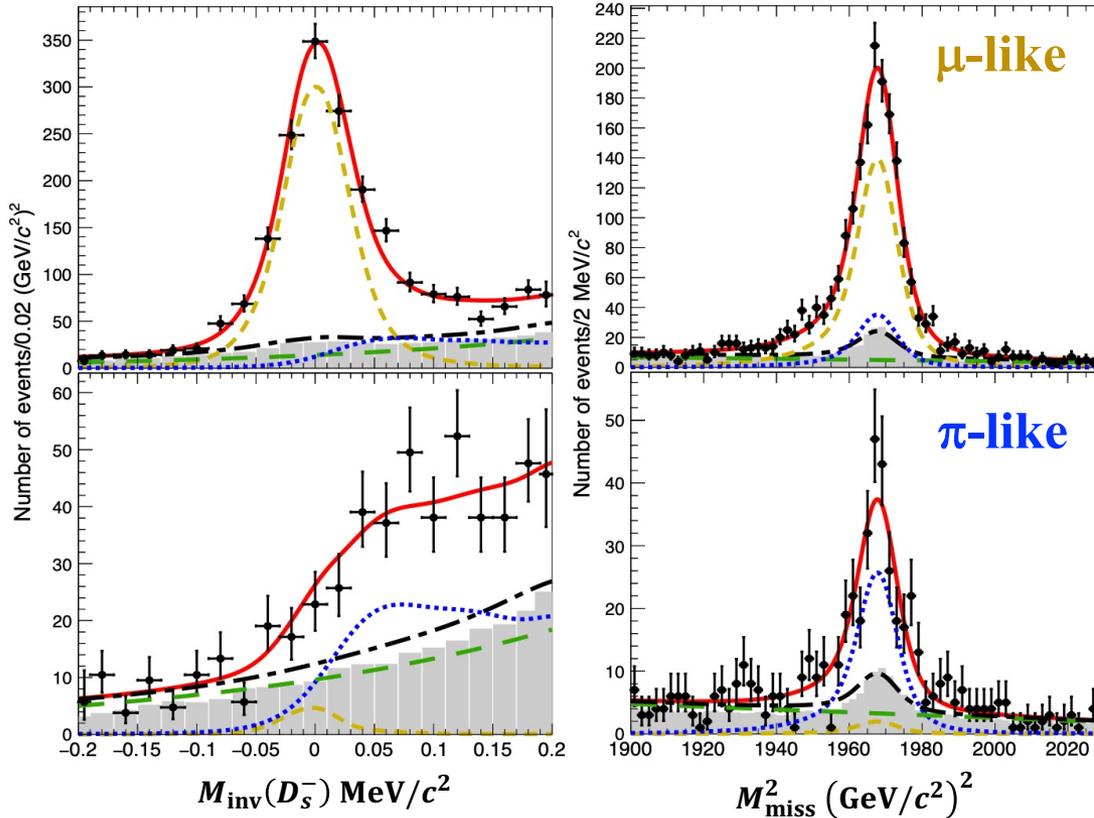
$$B_{D_s^+ \rightarrow \tau^+ \nu_\tau} = (5.29 \pm 0.25_{\text{stat.}} \pm 0.20_{\text{syst.}})\%$$

# $D_s^+ \rightarrow \tau^+ \nu_\tau$ via $\tau^+ \rightarrow \pi^+ \bar{\nu}_\tau$ and $D_s^+ \rightarrow \mu^+ \nu_\mu$

arXiv: 2102.11734 [hep-ex], accepted by PRD

- An unbinned **simultaneous** maximum likelihood fit to **two-dimensional** distributions

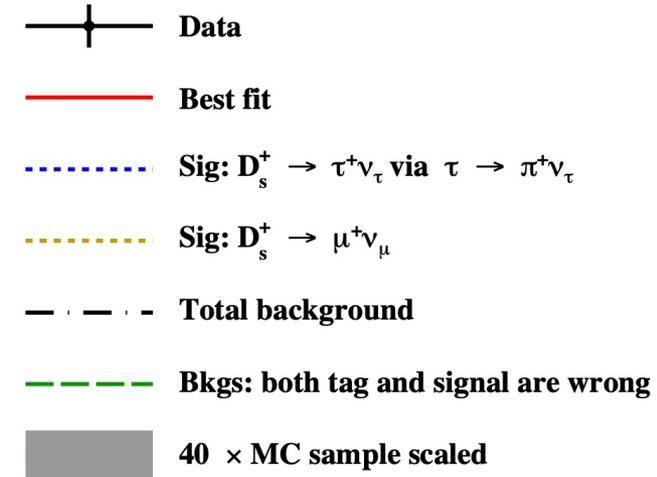
Only show @ 4.178 GeV



For all data samples

$$N_{D_s^+ \rightarrow \tau^+ \nu_\tau}^{\text{signal}} = 946^{+46}_{-45}$$

$$N_{D_s^+ \rightarrow \mu^+ \nu_\mu}^{\text{signal}} = 2198 \pm 55$$



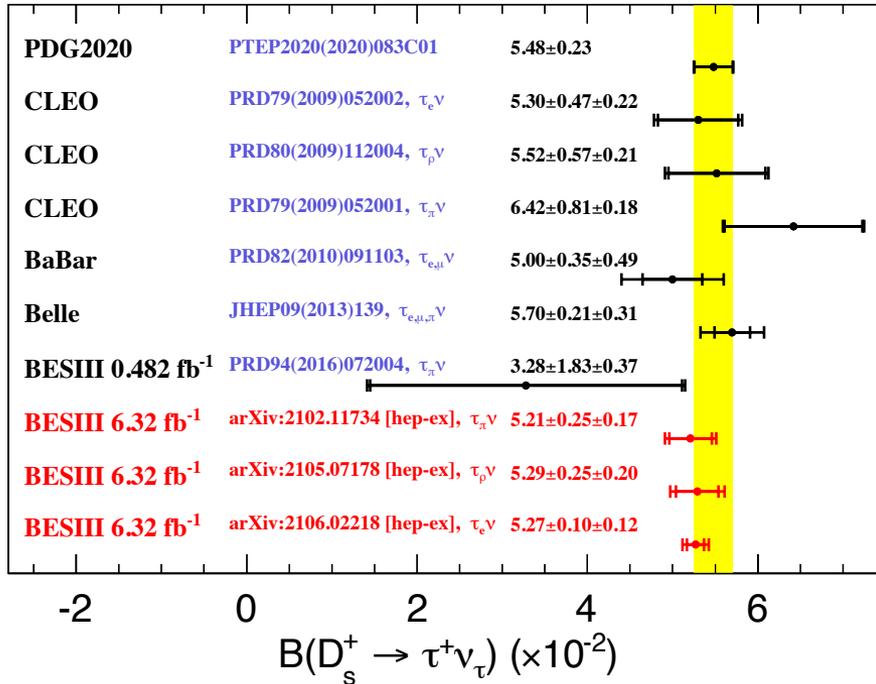
$$B(D_s^+ \rightarrow \tau^+ \nu_\tau) = (5.21 \pm 0.25_{\text{stat.}} \pm 0.17_{\text{syst.}}) \times 10^{-2}$$

$$B(D_s^+ \rightarrow \mu^+ \nu_\mu) = (5.35 \pm 0.13_{\text{stat.}} \pm 0.16_{\text{syst.}}) \times 10^{-3}$$

The most precise to date.

# Lepton flavor universality

- Combine results from **BESIII measurements** and PDG2020



$$R_{\tau/\mu} = \frac{\bar{\Gamma}(D_s^+ \rightarrow \tau^+ \nu_\tau)}{\bar{\Gamma}(D_s^+ \rightarrow \mu^+ \nu_\mu)} = \frac{m_{\tau^+}^2 \left(1 - \frac{m_{\tau^+}^2}{m_{D_s^+}^2}\right)^2}{m_{\mu^+}^2 \left(1 - \frac{m_{\mu^+}^2}{m_{D_s^+}^2}\right)^2}$$



$$= 9.67 \pm 0.34$$



$$9.75 \pm 0.01$$

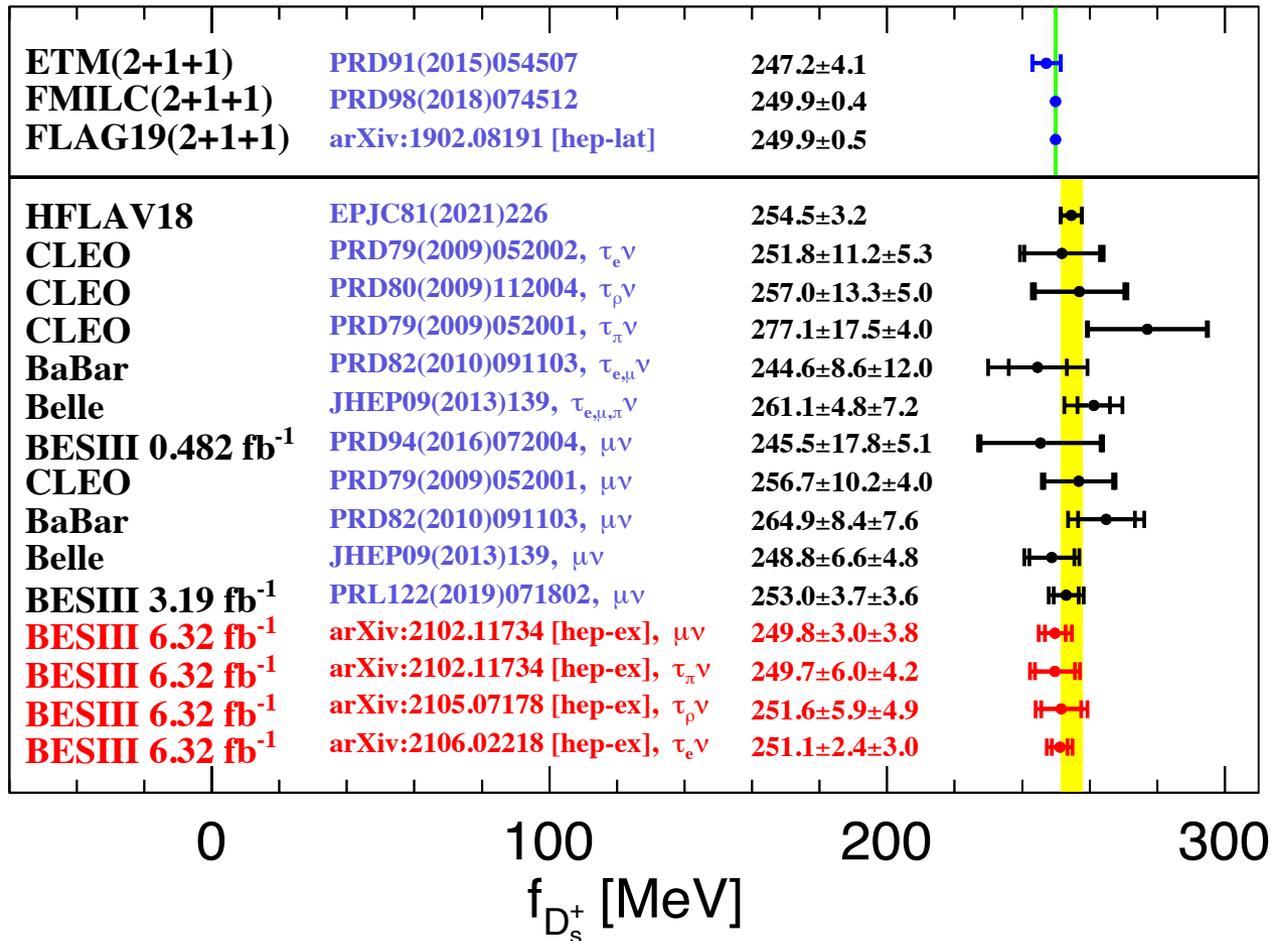
(SM prediction)

**No LFU violation** in  $\tau - \mu$  flavors with the current precision.

# Comparison of decay constant $f_{D_s^+}$

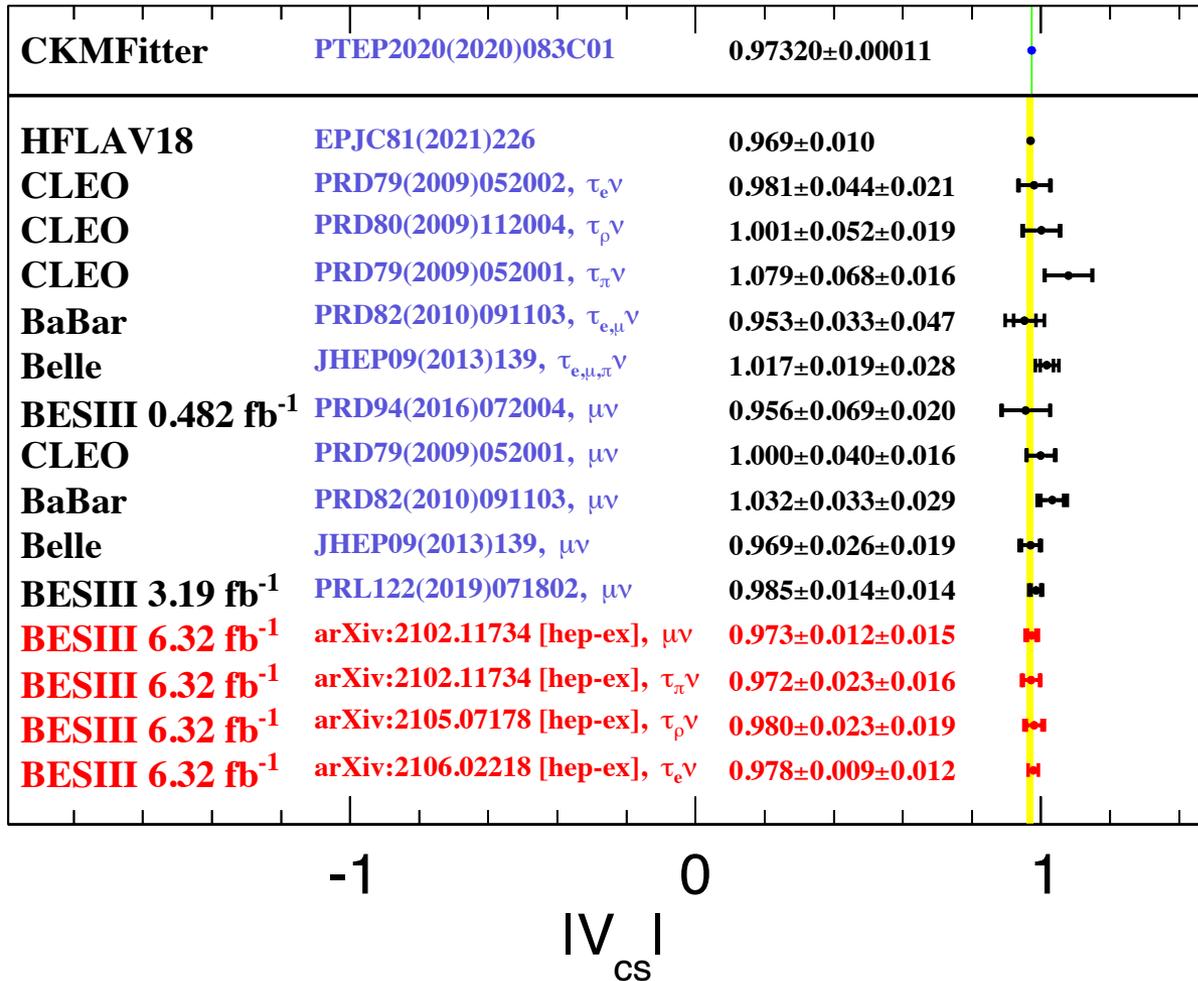
$$B(D_s^+ \rightarrow l^+ \nu_l) \propto [f_{D_s^+} |V_{cs}|]^2$$

- Input  $|V_{cs}| = 0.97320 \pm 0.00011$  from CKM global fit



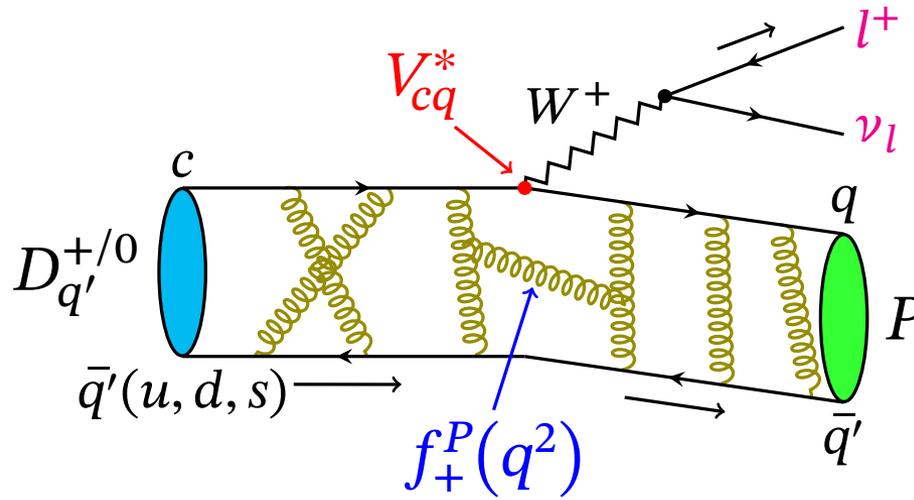
# Comparison of $|V_{cs}|$

- Input  $f_{D_s^+} = 249.9 \pm 0.5$  from LQCD calculations



# Semi-leptonic $D_{(s)}$ decay

In the SM



$$\frac{d\Gamma}{dq^2} = X \frac{G_F^2 p^3}{24\pi^3} |f_+(q^2)|^2 |V_{cd(s)}|^2, \quad (X = 1 \text{ for } K^-, \pi^-, \bar{K}^0, \eta^{(\prime)}; X = \frac{1}{2} \text{ for } \pi^0)$$

# $D^+ \rightarrow \eta \mu^+ \nu_\mu$

PRL124(2020)231801

No. of single tags:  $N_{ST}^{\text{signal}} = (1522.5 \pm 2.1) \times 10^3$

✓  $\eta \rightarrow \gamma\gamma$

✓ Unbinned fit to  $U_{\text{miss}}$

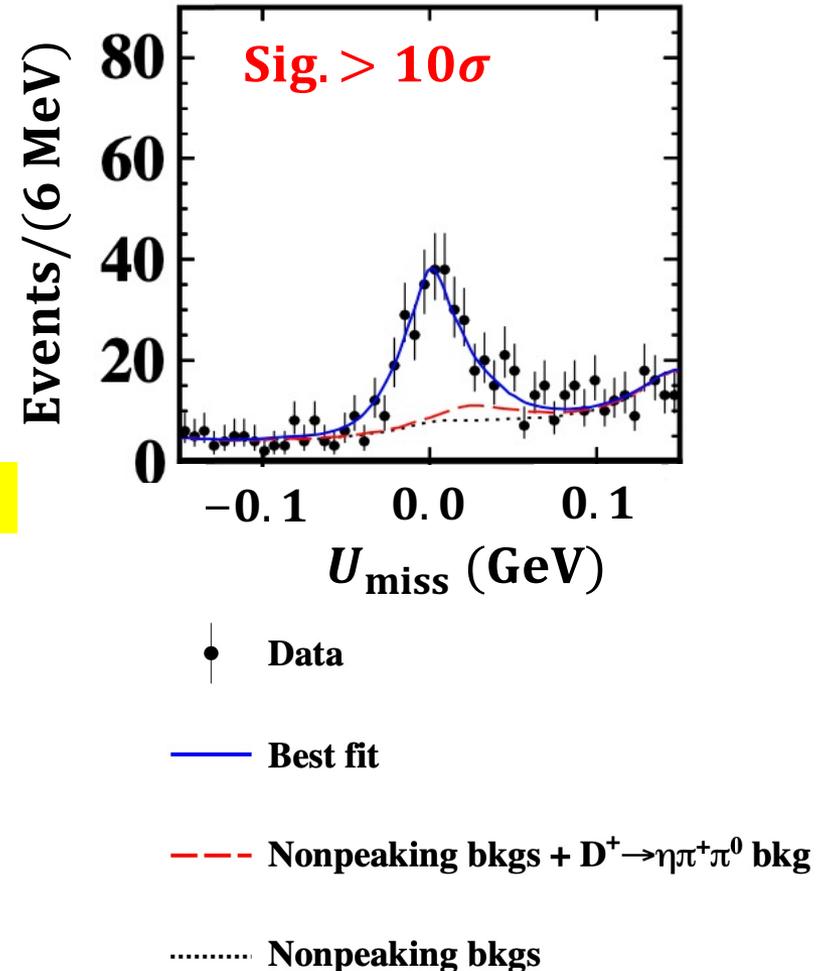
$N_{DT}^{\text{signal}} = 234 \pm 22$

$B_{D^+ \rightarrow \eta \mu^+ \nu_\mu} = (10.4 \pm 1.0_{\text{stat.}} \pm 0.5_{\text{syst.}}) \times 10^{-4}$

$$R_{\mu/e} = \frac{B_{D^+ \rightarrow \eta \mu^+ \nu_\mu}}{B_{D^+ \rightarrow \eta e^+ \nu_e}^{\text{PDG}}} = 0.91 \pm 0.13$$

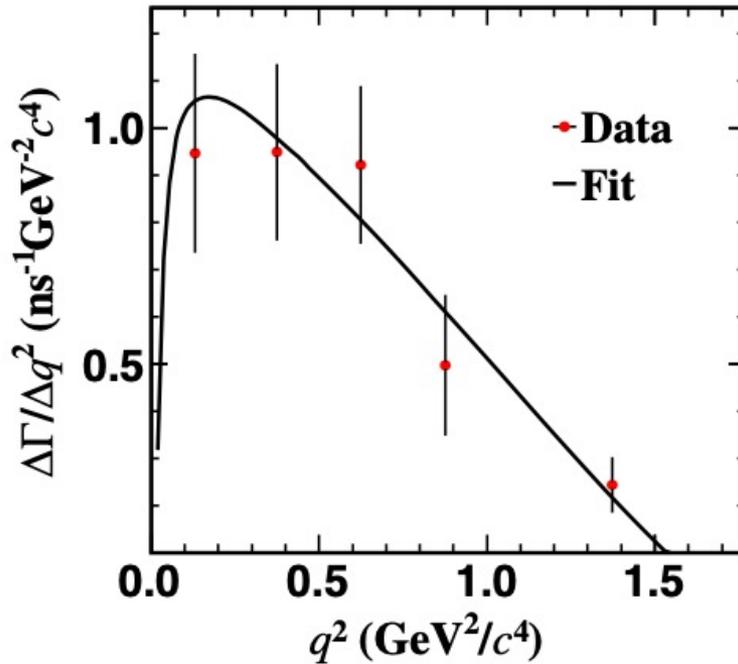
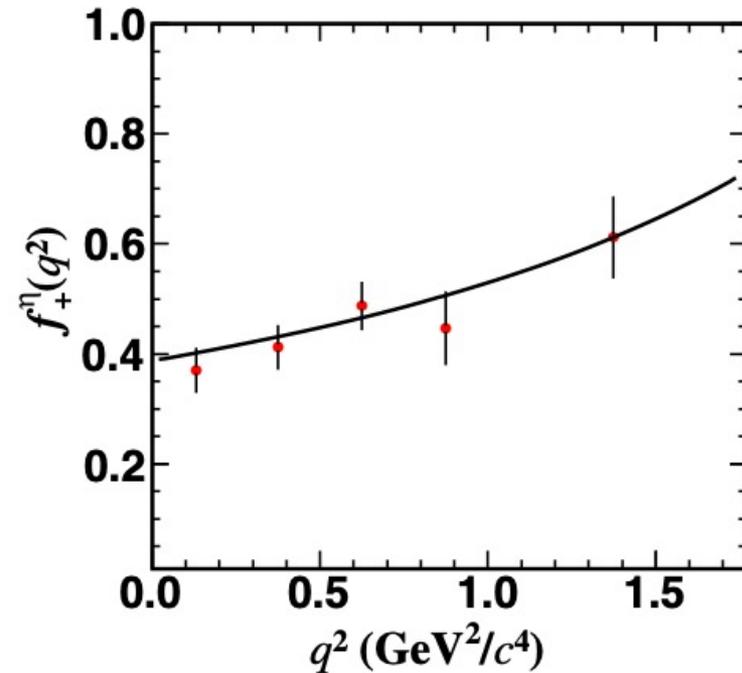
SM prediction:  $(0.97 - 1.00)$

No LFU violation within current sensitivity.



- First measurement on dynamics of  $D^+ \rightarrow \eta\mu^+\nu_\mu$  decay

Partial decay rates

Form factor  $f_+^\eta(q^2)$ 

$$f_+^\eta(0)|V_{cd}| = 0.087 \pm 0.008_{\text{stat.}} \pm 0.002_{\text{syst.}}$$

$$f_+^\eta(0) = 0.39 \pm 0.04_{\text{stat.}} \pm 0.01_{\text{syst.}}$$

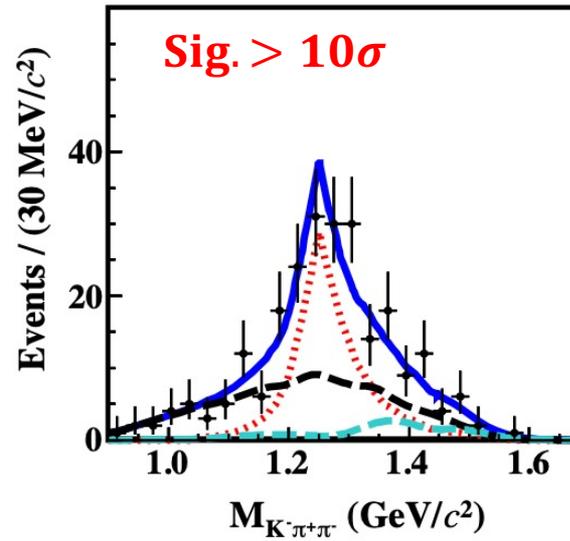
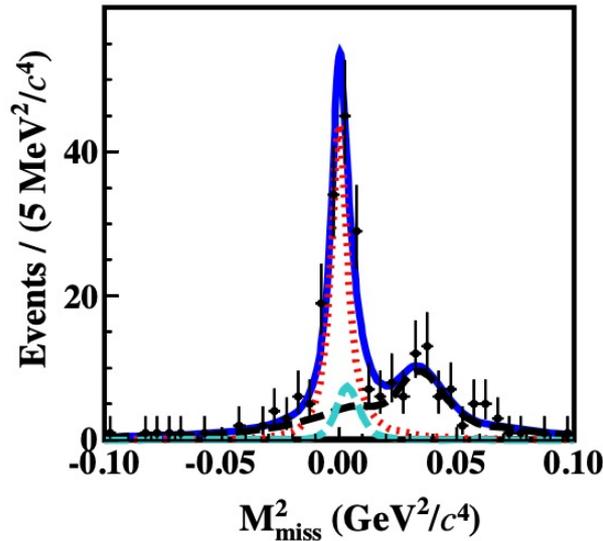
$$|V_{cd}| = 0.242 \pm 0.022_{\text{stat.}} \pm 0.006_{\text{syst.}} \pm 0.033_{\text{theory}}$$

$f_+^\eta(q^2)$  is parameterized by the two parameter series expansion.

# $D^0 \rightarrow K_1(1270)^- e^+ \nu_e$

arXiv: 2102.10850 [hep-ex]

- ✓  $K_1(1270)^- \rightarrow K^- \pi^+ \pi^-$
- ✓ **Two-dimensional** unbinned extended maximum-likelihood **simultaneous** fits shared with the **same value of**  $[B_{D^0 \rightarrow K_1(1270)^- e^+ \nu_e} \cdot B_{K_1(1270)^- \rightarrow X \rightarrow K^- \pi^+ \pi^-}]$ .



$$N_{DT}^{\text{signal}} = 109.0 \pm 12.5$$

- ✦ Data
- Best fit
- ⋯ Sig:  $D^0 \rightarrow K_1(1270)^- e^+ \nu_e$
- - - Peaking bkg:  $D^0 \rightarrow K^- \pi^+ \pi^+ \pi^-$
- - - Other bkg

$$B_{D^0 \rightarrow K_1(1270)^- e^+ \nu_e} = (1.09 \pm 0.13_{-0.13}^{+0.09} \pm 0.12_{\text{ex.}}) \times 10^{-3}$$

$$\frac{\Gamma_{D^0 \rightarrow K_1(1270)^- e^+ \nu_e}}{\Gamma_{D^+ \rightarrow \bar{K}_1(1270)^0 e^+ \nu_e}} = 1.20 \pm 0.02_{\text{stat.}} \pm 0.14_{\text{syst.}} \pm 0.04_{\text{ex.}}$$

Agrees with unity as predicted by isospin symmetry.

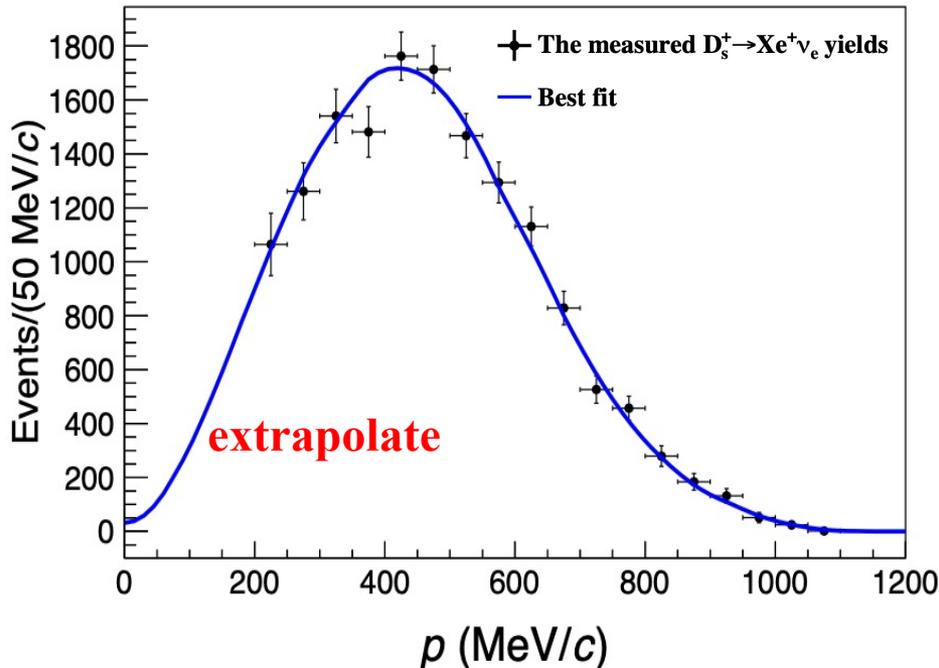
$$D_s^+ \rightarrow X e^+ \nu_e$$

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- ✓  $X$  means **inclusive** decays
- ✓ Sort recoil-side selected tracks into eighteen momentum ( $p_e$ ) bins for  $p_e > 200$  MeV/c
- ✓ The signal yield  $N_{D_s^+ \rightarrow X e^+ \nu_e}^{\text{signal}} = N_{p_e > 200 \text{ MeV/c}} + N_{p_e \leq 200 \text{ MeV/c}}$  (**extrapolate**)

$$N_{D_s^+ \rightarrow X e^+ \nu_e}^{\text{signal}} = 16648 \pm 326$$

$$B(D_s^+ \rightarrow X e^+ \nu_e) = (6.30 \pm 0.13_{\text{stat.}} \pm 0.10_{\text{syst.}}) \times 10^{-2}$$



$$\frac{\Gamma_{D_s^+ \rightarrow X e^+ \nu_e}}{\Gamma_{D^0 \rightarrow X e^+ \nu_e}} = 0.790 \pm 0.016_{\text{stat.}} \pm 0.020_{\text{syst.}}$$

- Consistent with the prediction of 0.813
- Supports the conclusion that the difference in the semileptonic decay widths of  $D_s^+$  ( $c\bar{s}$ ) and  $D^0$  ( $c\bar{u}$ ) mesons can be accounted for within the Standard model by **non-spectator interactions**.

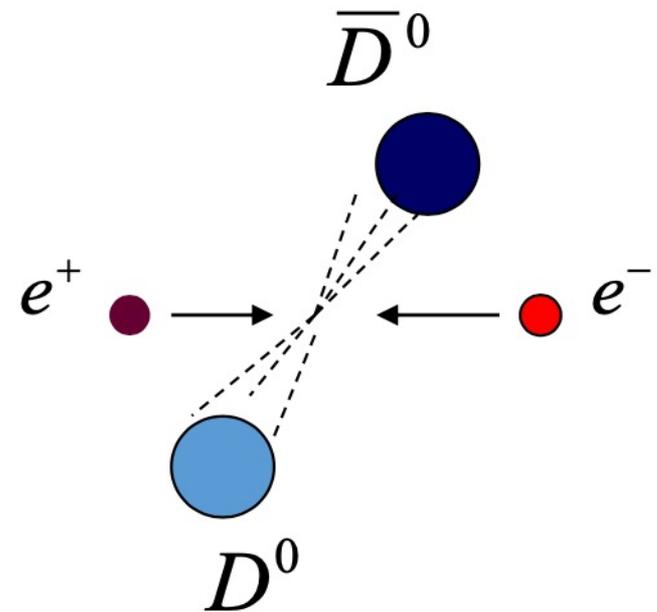
# Hadronic decays

The  $\psi(3770)$  has a  $C = -1$  quantum number and this is conserved in the strong decay in which two neutral  $D$  mesons are produced.

Hence, the two neutral mesons have an antisymmetric wave function. This also means that the two  $D$  mesons do not decay independently of one another.

The  $D^0\bar{D}^0$  pair will be a quantum-correlated state

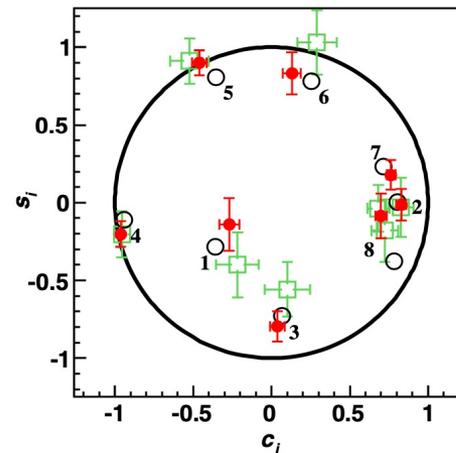
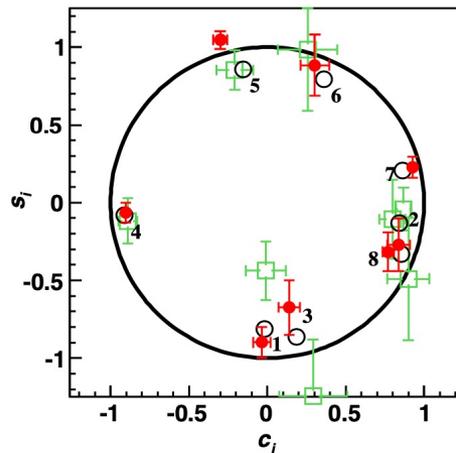
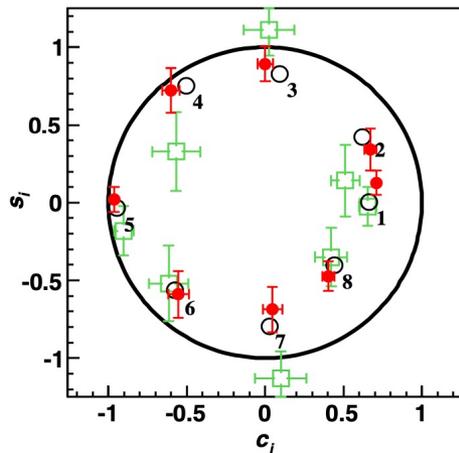
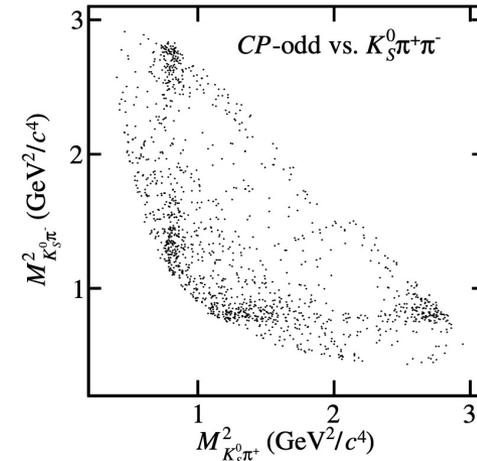
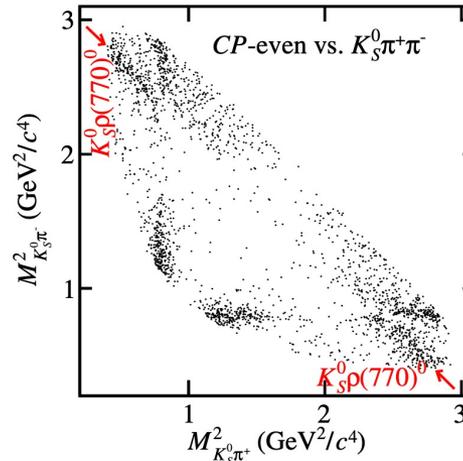
$$e^+e^- \rightarrow \psi(3770) \rightarrow D^0\bar{D}^0$$



# $D \rightarrow K_{S,L}^0 \pi^+ \pi^-$

PRL124(2020)241802; PRD101(2020)112002

- $\psi(3770) \rightarrow D^0 \bar{D}^0$  quantum correlation  $\rightarrow$  Directly measure the strong-phase difference  $\Delta\delta_D$  between  $D^0$  and  $\bar{D}^0$  decays  $\rightarrow$  The key inputs for a binned model-independent determination of the CKM angle  $\gamma/\phi_3$  with  $B$  decays.
- $c_i$  and  $s_i$  are the amplitude-weighted averages of  $\cos \Delta\delta_D$  and  $\sin \Delta\delta_D$  over each Dalitz plot bin.
- The estimated uncertainties on  $\gamma/\phi_3$ : between  $0.7^\circ$  and  $1.2^\circ$ .



$D \rightarrow K_S^0 \pi^+ \pi^-$

◆ This work

◆ CLEO results

○ Predictions from BABAR and Belle

Equal  $\Delta\delta_D$  binnings

Optimal  $\Delta\delta_D$  binnings

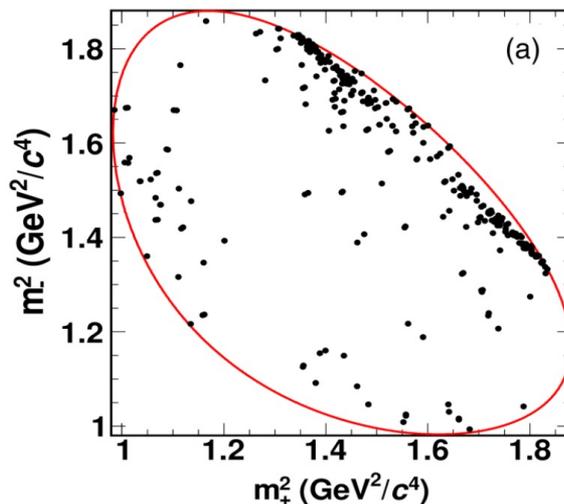
Modified optimal  $\Delta\delta_D$  binnings

# $D \rightarrow K_{S,L}^0 K^+ K^-$

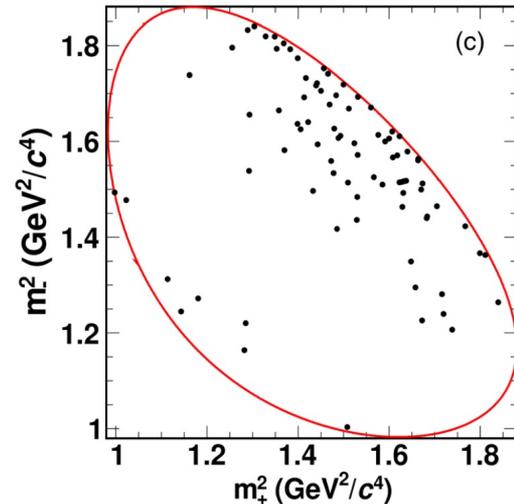
PRD102(2020)052008

- Equal  $\Delta\delta_D$  binnings scheme: with 2/3/4 pairs of bins.
- $m_{\pm}$ :  $m(K_S^0 K^{\pm})$ .
- **The estimated uncertainties on  $\gamma$  /  $\phi_3$ :  $2.3^\circ$ ,  $1.3^\circ$ , and  $1.3^\circ$  for 2, 3, 4 pairs of bins schemes.**

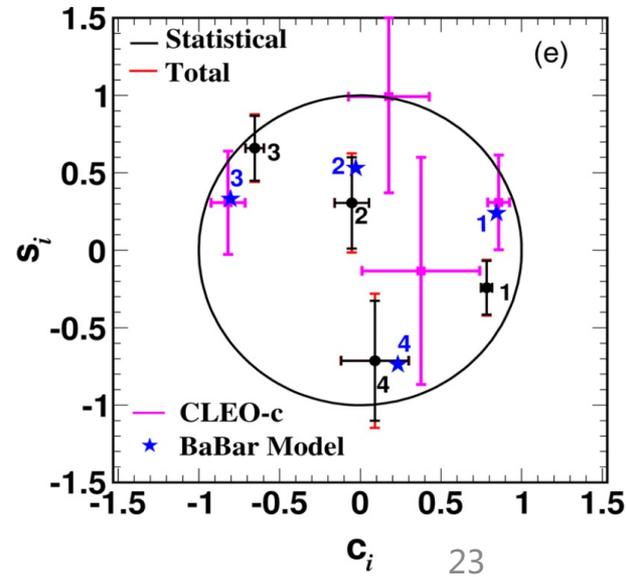
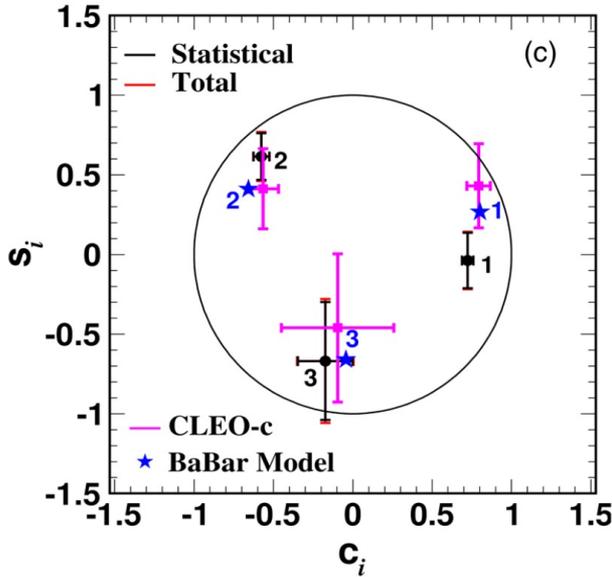
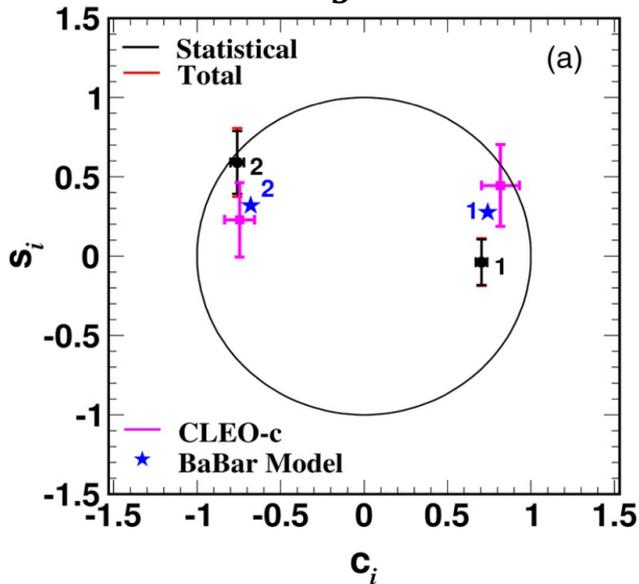
$CP$  – even vs.  $K_S^0 K^+ K^-$



$CP$  – old vs.  $K_S^0 K^+ K^-$



## $D \rightarrow K_S^0 K^+ K^-$



# $D \rightarrow K^+ \pi^+ \pi^+ \pi^-$ and $D \rightarrow K^- \pi^+ \pi^0$

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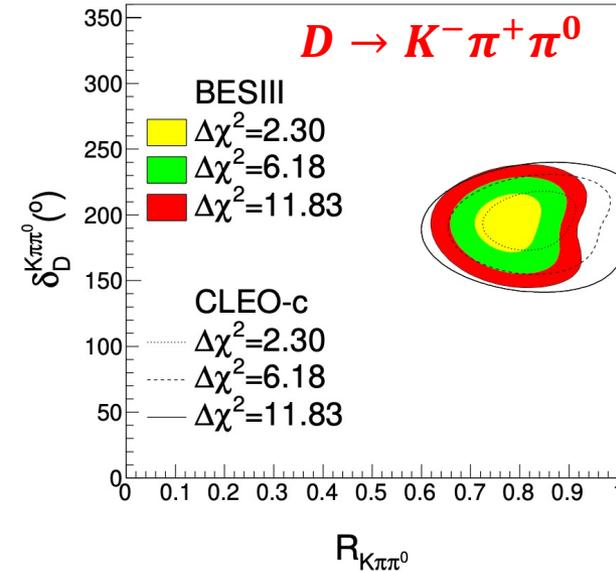
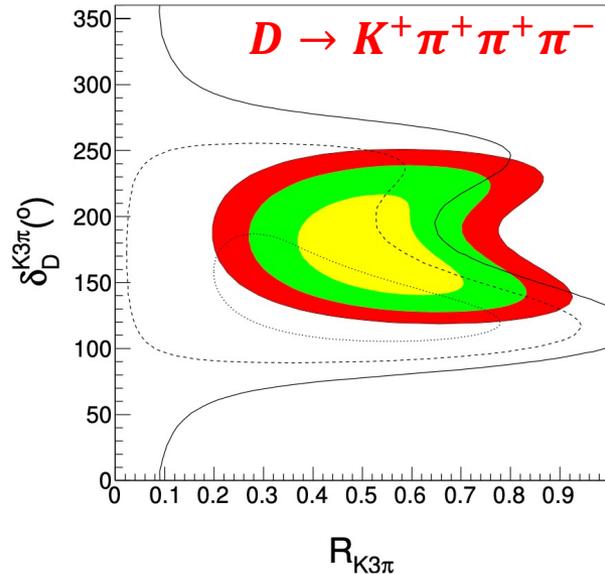
- A  $\chi^2$  fit : obtain the coherence factors ( $R$ ) and the average strong-phase differences ( $\delta_D$ )

$$R_{K3\pi} = 0.52^{+0.12}_{-0.10}$$

$$R_{K\pi\pi^0} = 0.78 \pm 0.04$$

$$\delta_D^{K3\pi} = (167^{+31}_{-19})^\circ$$

$$\delta_D^{K\pi\pi^0} = (196^{+14}_{-15})^\circ$$



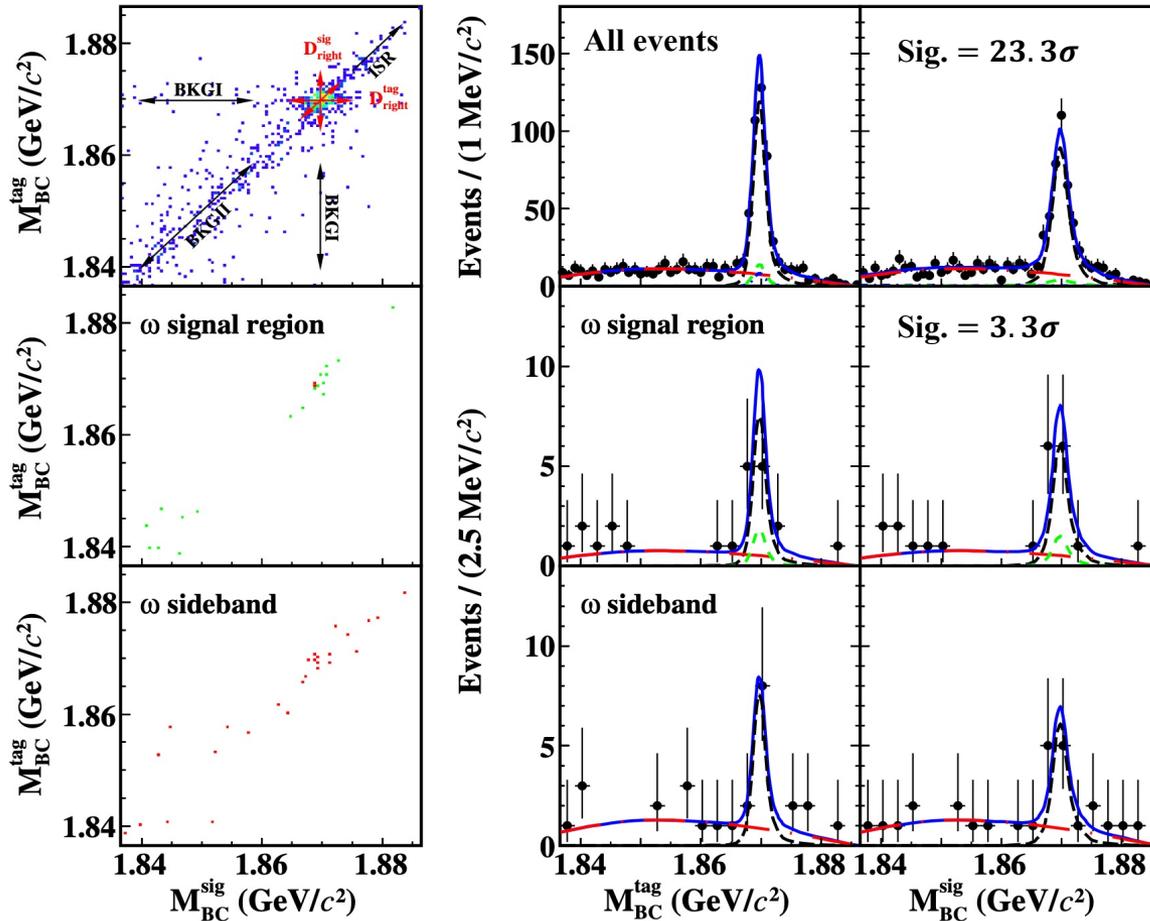
- $\Delta\chi^2$  scans:  $\Delta\chi^2 = 2.30, 6.18, 11.83$  intervals correspond to 68.3%, 95.4%, and 99.7% confidence levels in the 2D parameter space:

- Besides, the phase space of  $D \rightarrow K^+ \pi^+ \pi^+ \pi^-$  is divided into four bins. Based on this results, **the estimated uncertainty on  $\gamma/\phi_3$  is around  $6^\circ$ .**

# $D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0$

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- The naive expectation for the doubly Cabibbo-suppressed (DCS) decay rate relative to its Cabibbo-favored (CF) counterpart  $R = B(\text{DCS})/B(\text{CF})$  is of the order  $\tan^4 \theta_C$  ( $\sim 0.29\%$ ) level, where  $\theta_C$  is the Cabibbo mixing angle.
- 2D fits for all events; simultaneous 2D fits for  $\omega$  events



$$N_{\text{DT}}^{D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0} = 350 \pm 22$$

$$N_{\text{DT}}^{D^+ \rightarrow K^+ \omega} = 9.2^{+4.0}_{-3.4}$$

- Data
- Best fit
- Signal
- · - · BKG I: only one D is right
- · - · BKG II:  $e^+e^- \rightarrow q\bar{q}$
- · - · Peaking bkg:

For all events,  $D^+ \rightarrow K^+ \pi^+ K^- (\rightarrow \pi^+ \pi^0)$   
 +  $D^+ \rightarrow K^+ K_S^0 (\rightarrow \pi^+ \pi^-) \pi^0$

For  $\omega$  events,  $D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0$

$$B_{D^+ \rightarrow K^+ \omega} = (5.7_{-2.1}^{+2.5}{}_{\text{stat.}} \pm 0.2_{\text{syst.}}) \times 10^{-5}$$

$$B_{D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0}^* = (1.13 \pm 0.08_{\text{stat.}} \pm 0.03_{\text{syst.}}) \times 10^{-3}$$

\*Remove contributions from  $\eta/\omega/\phi \rightarrow \pi^+ \pi^- \pi^0$

- The ratio  $B(\text{DCS})/B(\text{CF})$  :

$$\frac{B_{D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0}^*}{\bar{B}_{D^+ \rightarrow K^- \pi^+ \pi^+ \pi^0}^{\text{PDG}}} = (1.81 \pm 0.15)\% = (6.28 \pm 0.52) \tan^4 \theta_C$$

This ratio is **significantly larger** than the values (0.21~0.58)% from other DCS decays, which implies that there is a massive isospin symmetry violation

- The asymmetry:

$$A_{CP}^{D^\pm \rightarrow K^\pm \pi^\pm \pi^\mp \pi^0} = \frac{B_{D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0} - B_{D^- \rightarrow K^- \pi^- \pi^+ \pi^0}}{B_{D^+ \rightarrow K^+ \pi^+ \pi^- \pi^0} + B_{D^- \rightarrow K^- \pi^- \pi^+ \pi^0}} = (-0.04 \pm 0.06_{\text{stat.}} \pm 0.01_{\text{syst.}})$$

No evidence for CP violation is found.

# Summary

👉 With  $2.93 \text{ fb}^{-1}$  @ 3.773 GeV and  $6.32 \text{ fb}^{-1}$  from 4.178-4.226 GeV data samples, BESIII have studied

- pure and semi-leptonic  $D_{(s)}$  decays, which provide precision calibration of LQCD and precision measurements of CKM matrix elements.
- charm hadronic decays, which are key labs to understand non-perturbative QCD, and provide important inputs to model-independent determination of CKM angle  $\gamma/\phi_3$  and charm mixing/CPV.

👉 In the near future, BESIII will collect  $20 \text{ fb}^{-1}$  @ 3.773 GeV data sample, and another  $3 \text{ fb}^{-1}$  @ 4.178 GeV[1], the single precisions will be further improved → a new era of precision charm physics.

# Backup

- **Form factor  $f_+(0)$  with input  $|V_{cs}|^{\text{CKMfitter}}$**

- **Single pole model**

$$f_+(q^2) = \frac{f_+(0)}{1 - q^2/M_{\text{pole}}^2}$$

- **ISGW2 model**

$$f_+(q^2) = f_+(q_{\text{max}}^2) \left( 1 + \frac{r^2}{12} (q_{\text{max}}^2 - q^2) \right)^{-2}$$

- **Modified pole model**

$$f_+(q^2) = \frac{f_+(0)}{\left( 1 - \frac{q^2}{M_{\text{pole}}^2} \right) \left( 1 - \alpha \frac{q^2}{M_{\text{pole}}^2} \right)}$$

- **Series expansion**

$$f_+(t) = \frac{1}{P(t)\Phi(t, t_0)} a_0(t_0) \left( 1 + \sum_{k=1}^{\infty} r_k(t_0) [z(t, t_0)]^k \right)$$