Zero-range potentials: Delta-like barrier versus self-adjoint extension

Yuri V. Grats, Pavel Spirin

Moscow State U.

20th Lomonosov Conference
August 19-25, 2021, Moscow, RF

- *Y.V. Grats*, Phys. Atom. Nucl. **82**, (2019) no.2, 153
- *Y.V. Grats, P.Spirin*, Universe **7**, (2021) no.5, 127

Content:

Cosmic string: motivation

1st PT order vs NP in literature:

$$\Delta T_{\mu\nu} \propto \xi(\xi - 1/6)$$

Field eqn.

$$\sqrt{-g}\,[\Box - \xi R]\,G^F_\xi(x,x') = -\delta^d(x-x')$$

Potential:

$$\gamma(r) = \sqrt{-g}\,\xi R = 4\pi(1-\beta)\xi\,\delta^2(\boldsymbol{\rho})$$

Limit: $(1-\beta) \to 0 \quad \xi \to \infty \quad (1-\beta)\xi = \text{const}$

$$\left[\partial^2 + \lambda\,\delta^2(\boldsymbol{\rho})\right] G^F_\lambda(x,x') = -\delta^d(x-x') \qquad \boldsymbol{\rho} = (x,y)$$

$$\left[\partial^2 + \lambda\,\delta^3(\boldsymbol{r})\right] G^F_\lambda(x,x') = -\delta^d(x-x'), \qquad \boldsymbol{r} = (x,y,z)$$

Green's function: perturbation theory

Field equation:

$$\mathcal{L}(x,\partial)\, G^F(x,x') = -\delta^d(x-x')$$

Operator perturbation: $\mathfrak{L} = \mathfrak{L}_0 + \delta\mathfrak{L}$

$$\mathfrak{L}\mathcal{G} = -1 \qquad \mathcal{G} = -\mathfrak{L}^{-1} \qquad \mathcal{G}_0 = -\mathfrak{L}_0{}^{-1}$$

Solution as series

$$\mathcal{G} = \Big[-\mathfrak{L}_0\,(1-\mathcal{G}_0\delta\mathfrak{L})\Big]^{-1} = \mathcal{G}_0 + \mathcal{G}_0\,\delta\mathfrak{L}\,\mathcal{G}_0 + \,\ldots$$

For zero-order equation

$$\mathcal{L}_0(x,\partial) = \partial^2 \qquad G_0^F(x-x') = \int \frac{d^dp}{(2\pi)^d}\,\frac{e^{ip\,(x-x')}}{p^2+i\epsilon}$$

Operator deviation:

$$\delta\mathcal{L}(x,\partial) = \mathcal{U}(x)$$

Renormalized VEVs

Dimensional regularization: $D = d - 2\delta$

$$G^F_{\rm ren}(x,x\,|\,d) = \lim_{\delta\to 0} G^F_{\rm reg}(x,x\,|\,D)$$

Renormalization:

$$G^F_{\rm ren}(x,x\,|\,d) = \lim_{\delta\to 0}\left[G^F_{\rm reg}(x,x\,|\,D) - G^F_{\rm div}(x,x\,|\,D)\right]$$

Renormalized VEV $\langle\varphi^2(x)\rangle$:

$$\langle\varphi^2(x)\rangle_{\rm ren} = i\,G^F_{\rm ren}(x,x\,|\,d)$$

Renormalized energy-momentum tensor:

$$\left\langle T^\nu_\mu\right\rangle_{\rm ren} = i\,D^\nu_\mu\,G^F_{\rm ren}(x,x'\,|\,d)\Big|_{x'=x}$$

$$D^\nu_\mu = (1-2\xi)\,\partial^\nu\partial'_\mu - \left(\frac{1}{2}-2\xi\right)\delta^\nu_\mu\partial^\lambda\partial'_\lambda - \xi\left(\nabla^\nu\partial_\mu + \nabla'^\nu\partial'_\mu\right)$$

Static potentials

Fourier-transform: $\mathcal{U}(q) = 2\pi\,\delta(q^0)\,\mathcal{U}(\boldsymbol{q})$

The generic form: $(\tilde{q}^\mu = (0, \boldsymbol{q}))$

$$\text{VEV} = -i\int\frac{d\,\boldsymbol{q}}{(2\pi)^3}\,e^{i\boldsymbol{q}\,\boldsymbol{x}}\,\mathcal{U}(\boldsymbol{q})\int\frac{d^4p}{(2\pi)^4}\,\frac{Q}{[p^2+i\varepsilon][(p+\tilde{q})^2+i\varepsilon]}$$

VEV	Q
$\langle\varphi^2\rangle$	1
$\langle T_{00}\rangle$	$p_0^2 + [2\xi - 1/2]\,p\cdot\tilde{q}$
$\langle T_{0i}\rangle$	$(\xi - 1)\,p_0\tilde{q}_i - p_0p_i$
$\langle T_{ij}\rangle$	$(\xi-1)\,\tilde{q}_ip_j - p_ip_j + [2\xi-1/2]\,p\cdot\tilde{q}\,\delta_{ij} - \xi(p_i\tilde{q}_j + \tilde{q}_i\tilde{q}_j)$

$$\left\langle\phi^2(r)\right\rangle_{\text{ren}} = \frac{1}{32\pi^3}\int d\boldsymbol{r}'\,\frac{\mathcal{U}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

Renormalized EMT:

$$\langle T_{\mu\nu}(\boldsymbol{r})\rangle_{\rm ren} = \frac{3}{32\pi^3}\left(\xi - \frac{1}{6}\right)\int d^3r' \frac{\mathcal{U}(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^5} Z_{\mu\nu}$$

$$Z_{00} = 2 \qquad Z_{ij} = 5\frac{x_i x_j - 2x_{(i}x'_{j)} + x'_i x'_j}{|\boldsymbol{r} - \boldsymbol{r'}|^2} - 3\delta_{ij}$$

Conservation: $\partial^\nu T_{\mu\nu} = \partial^i T_{ij} \propto \partial^i\left(Z_{ij}|\boldsymbol{r} - \boldsymbol{r'}|^{-5}\right) = 0$

Spherically-symmetric potential: $r' < R \ll r$

$$\langle \phi^2(r)\rangle_{\rm ren} = -\frac{1}{8\pi^2 r}\int_0^R dr' \frac{r'^2\,\mathcal{U}(r')}{r^2 - r'^2}$$

$$\langle T^0_0(r)\rangle_{\rm ren} = \frac{1}{4\pi^2 r}\left(\xi - \frac{1}{6}\right)\int_0^R dr'\,\mathcal{U}(r')\frac{r'^2(3r^2 + r'^2)}{(r^2 - r'^2)^3}$$

Trying $\delta^3(\boldsymbol{r})$ potential as perturbation

Klein-Gordon eqn: $[\lambda] = \mathrm{cm}$ $|\lambda| \ll$ "r_0"

$$\left[\partial^2 + \lambda\,\delta^3(\boldsymbol{r})\right]\phi(t,\boldsymbol{r}) = 0$$

Splitting of the field operator:

$$\mathcal{L}_0(x) = \partial^2\,, \qquad \delta\mathcal{L}(x) = \lambda\,\delta^3(\boldsymbol{r}) \qquad \text{“}\delta\mathcal{L}\text{“} \ll \text{“}\mathcal{L}_0$$

Renormalized VEV:

$$\langle\phi^2(r)\rangle_{\rm ren} = -\frac{\lambda}{32\pi^3 r^3}$$

Renormalized enegry-momentum tensor:

$$\left\langle T_\mu^\nu(t,r,\vartheta,\varphi)\right\rangle_{\rm ren} = \frac{\lambda}{64\pi^3 r^5}(6\xi - 1)\,\mathrm{diag}\left(1,-1,\frac{3}{2},\frac{3}{2}\right)$$

ξ and λ are independent!

$\delta^3(\boldsymbol{r})$ interaction via self-adjoint extension

$$\left[\partial_t^2 - \Delta + \lambda\,\delta^3(\boldsymbol{r})\right]\phi(t,\boldsymbol{r}) = 0$$

Factorization: $\quad H \equiv -\Delta + \lambda\,\delta^3(\boldsymbol{x})$

$$\phi^{\pm}(t,\boldsymbol{r}) = \mathrm{e}^{\mp i\omega t} u_\omega(\boldsymbol{r})\,, \qquad H u_\omega = \omega^2 u_\omega$$

Hamiltonian decomposition

$$H \to \dot{H} = \bigoplus_{l=0}^{\infty} \dot{H}_l \underbrace{\bigotimes \mathbb{I}}_{\text{angular}} \qquad \dot{H}_l = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2}$$

Self-adjointness

$$\mathbb{R}^3 \to \mathbb{R}^3 \backslash \mathbf{0}\,, \qquad \dot{H}_0 \to \dot{H}_{0,\alpha} \qquad u_\omega \to u_{\omega,\alpha}$$

$+$ initial condition depending on parameter α

$\alpha \to +\infty$ as $\lambda \to 0$ (free Hamiltonian)

Eigenfunctions and eigenvalues

Attraction ($\alpha < 0$):

$$\text{continuous}: \quad u_{\omega,\alpha} = \frac{1}{2\pi r}\frac{\sin(\omega r + \delta)}{\sqrt{\omega}} \qquad E = \omega^2$$

$$\text{discrete}: \quad u_{\alpha} = \frac{\sqrt{|\alpha|}}{r}\,\mathrm{e}^{-4\pi|\alpha|r} \qquad E = -(4\pi\alpha)^2$$

Repulsion ($\alpha > 0$):

$$\text{continuous}: \quad u_{\omega,\alpha} = \frac{1}{2\pi r}\frac{\sin(\omega r + \delta)}{\sqrt{\omega}} \qquad \tan\delta = \frac{\omega}{4\pi\alpha}$$

no negative energy modes!

Hadamard function $D^{(1)}(x, x') = \langle \phi(x)\phi(x') + \leftrightarrow \rangle/2$

$$D^{(1)} = \mathrm{Re}\int d\omega\, \mathrm{e}^{i\omega(t'-t)}\left[u^*_{\omega,\alpha}(x')\,u_{\omega,\alpha}(x) + \sum_{l>0,m} u^*_{lm}(x')\,u_{lm}(x)\right]$$

Renormalized Hadamard function

$$D^{(1)}_{\alpha} = \mathrm{Re}\int d\omega\, \mathrm{e}^{i\omega(t'-t)} \left[u^*_{\omega,\alpha}(x')\, u_{\omega,\alpha}(x) - u^*_{\omega,\infty}(x')\, u_{\omega,\infty}(x) \right]$$

Scalar-field VEV

$$\langle \phi^2 \rangle_{\mathrm{ren}} = D^{(1)}_{\alpha}(x,x) = \frac{\mathcal{J}(\beta)}{4\pi^2 r^2}\,,$$

where

$$\mathcal{J}(\beta) = \int\limits_0^{\infty} dz\, \frac{\sin\beta z + z\cos\beta z}{1+z^2} \qquad \beta = 8\pi\alpha r$$

Similar results: *D.Fermi, L.Pizzocchero, Symmetry (2018) 10, 38*

$$\mathcal{J}(\beta) = \mathrm{e}^{-\beta} E_1(\beta)\,, \qquad E_1(\beta) \equiv \int\limits_{\beta}^{\infty} \frac{\mathrm{e}^{-t}}{t}\, dt$$

Renormalized EMT

Introducing $\mathcal{J}_k(\beta) \equiv \beta^k d^k \mathcal{J}(\beta)/d\beta^k$

$$\langle T_t^t \rangle_{\rm ren} = \frac{1}{4\pi^2 r^4}\left[\left(\frac{1}{2} - 2\xi\right)\mathcal{J} + \left(2\xi - \frac{1}{2}\right)\mathcal{J}_1 - \xi\mathcal{J}_2\right]$$

$$\langle T_r^r \rangle_{\rm ren} = \frac{1}{4\pi^2 r^4}\left[\left(4\xi - \frac{1}{2}\right)\mathcal{J} + \left(\frac{1}{2} - 2\xi\right)\mathcal{J}_1\right]$$

$$\langle T_\theta^\theta \rangle_{\rm ren} = \frac{1}{4\pi^2 r^4}\left[\left(\frac{1}{2} - 4\xi\right)\mathcal{J} + \left(3\xi - \frac{1}{2}\right)\mathcal{J}_1 + \left(\frac{1}{4} - \xi\right)\mathcal{J}_2\right]$$

Total trace:

$$\mathrm{Sp}\langle T_\mu^\nu \rangle_{\rm ren} = (1 - 6\xi)\,\frac{2\mathcal{J}(\beta) - 2\mathcal{J}_1(\beta) + \mathcal{J}_2(\beta)}{8\pi^2 r^4}$$

Splitting:

$$\langle T_\mu^\nu \rangle_{\rm ren} = \tilde{T}_\mu^\nu + \left(\xi - \frac{1}{6}\right)\bar{T}_\mu^\nu(\xi)\,,$$

Small αr

For $\alpha r \ll 1$ $\quad E_1(x) = -\ln x - \gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k \cdot k!}$

Scalar-field VEV

$$\langle \phi^2 \rangle_{\text{ren}} = \frac{|\ln(8\pi \mathrm{e}^{\gamma} \alpha r)|}{4\pi^2 r^2},$$

Energy-momentum-tensor:

$$\left\langle T_\mu^\nu(t, r, \vartheta, \varphi) \right\rangle_{\text{ren}} = \frac{|\ln \alpha r|}{2\pi^2 r^4} \left[\left(\xi - \frac{1}{6} \right) \mathrm{diag}\left(1, -2, 2, 2\right) - - \frac{1}{12} \mathrm{diag}\left(1, 1, -1, -1\right) \right]$$

Conservation and trace:

$$\partial_\nu \langle T_\mu^\nu \rangle_{\text{ren}} = 0 \qquad \mathrm{Sp} \langle T \rangle_{\text{ren}} \Big|_{\xi = \xi_4} = 0$$

Large αr

For $\alpha r \gg 1$

$$\langle \phi^2 \rangle_{\rm ren} = \frac{1}{32\,\pi^3 \alpha r^3}\left[1 + \mathcal{O}\left(\frac{1}{\alpha r}\right)\right]$$

Naive relation with λ:

$$\lambda - = \frac{1}{\alpha} + \mathcal{O}\left(\frac{1}{\alpha^2 r_0}\right) \longrightarrow -\frac{1}{\alpha}$$

Energy-momentum-tensor:

$$\langle T_\mu^\nu(t, r, \vartheta, \varphi)\rangle_{\rm ren} = \frac{(1-6\xi)}{2^6 \pi^3 r^5\,\alpha}\,\mathrm{diag}\left(1, -1, \frac{3}{2}, \frac{3}{2}\right) + \ldots$$

Conformal tensor:

$$\left\langle \tilde{T}_\mu^\nu(t, r, \theta, \varphi)\right\rangle_{\rm ren} = \frac{1}{512\alpha^2\pi^4 r^6}\,\mathrm{diag}\left(1, \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right) + \ldots$$

Correspondence of two approaches

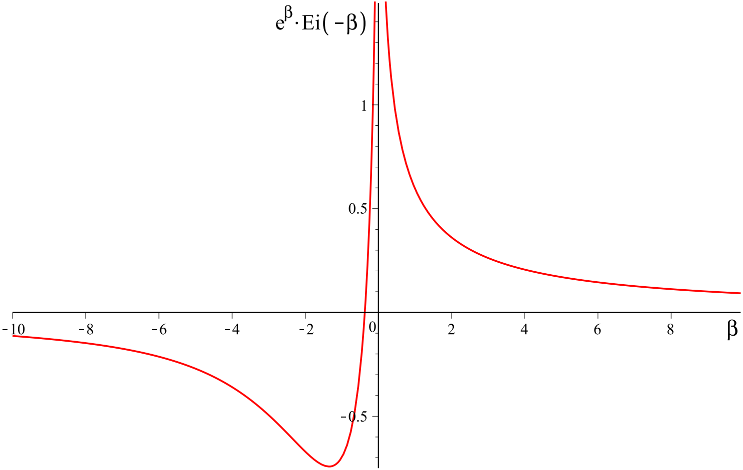
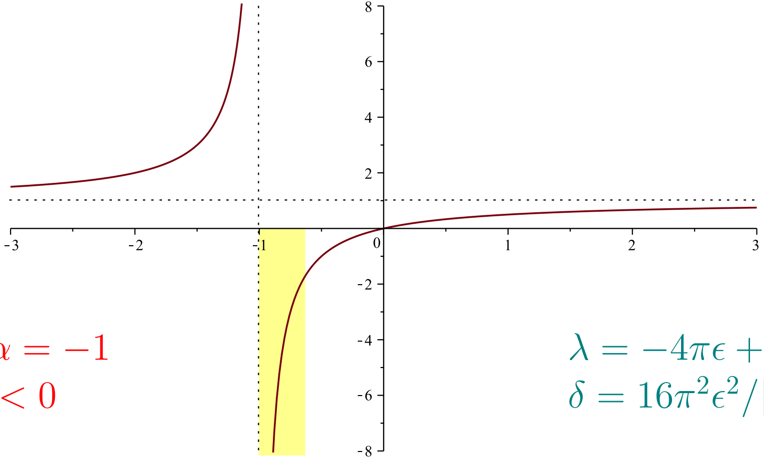


Figure: Plot of J for all real arguments

$$\beta > 0 \qquad \alpha > 0$$

Renormalization

$$\frac{1}{\lambda_{\rm ren}} = \frac{1}{\lambda} + \frac{1}{4\pi\epsilon}\,, \qquad \epsilon \to 0^+ \qquad \lambda_{\rm ren} = \frac{4\pi\lambda}{\lambda + 4\pi\epsilon}\,\epsilon$$



$$\lambda_{\rm ren}\alpha = -1$$
$$\lambda_{\rm ren} < 0$$

$$\lambda = -4\pi\epsilon + \delta$$
$$\delta = 16\pi^2\epsilon^2/|\lambda_{\rm ren}|$$

Figure: Plot of $\lambda_{\rm ren}/4\pi\epsilon$ versus $\lambda/4\pi\epsilon$

Conclusions

- Static-potential-induced vacuum polarization of a massless scalar field is computed within the PT framework;
- Zero-order consists of tadpoles and vanishes in regularization;
- The Laplacian's SAE gives the exact result for the repulsive 3D Aharonov–Bohm barrier
- In order to have a correspondence, we have to impose the coupling renormalization
- In the PT-framework formulae one also has to substitute λ by λ_{ren}
- The case of 2D point potential (the real cosmic-string background), is of the future interest.

Thank you!