Twentieth Lomonosov Conference on Elementary Particle Physics

August 19–25, 2021, Moscow State University, Moscow

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NSVZ β-function and NSVZ scheme with the higher covariant derivative regularization in the non-Abelian case

We will investigate the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function.

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

It relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories,

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j (\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$

$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

The exact NSVZ β-functions for MSSM

The NSVZ equations can be written for phenomenologically interesting theories, including theories with multiple gauge couplings.

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761.

For instance, the all-order exact MSSM β-functions are written as

$$\frac{\beta_3(\alpha,\lambda)}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3+\sum_{I=1}^{3}\Big(\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{D_I}(\alpha,\lambda)\Big)\Big];$$

$$\frac{\beta_2(\alpha,\lambda)}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1+\sum_{I=1}^{3}\Big(\frac{3}{2}\gamma_{Q_I}(\alpha,\lambda)+\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)$$
$$+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11+\sum_{I=1}^{3}\Big(\frac{1}{6}\gamma_{Q_I}(\alpha,\lambda)+\frac{4}{3}\gamma_{U_I}(\alpha,\lambda)+\frac{1}{3}\gamma_{D_I}(\alpha,\lambda)$$
$$+\frac{1}{2}\gamma_{L_I}(\alpha,\lambda)+\gamma_{E_I}(\alpha,\lambda)\Big)+\frac{1}{2}\gamma_{H_u}(\alpha,\lambda)+\frac{1}{2}\gamma_{H_d}(\alpha,\lambda)\Big]$$

and correctly reproduce the (scheme-independent) two-loop contributions.

The exact NSVZ β-functions for the flipped $SU(5)$ model

As another example we consider the flipped $SU(5)$ Grand Unification Theory.

S. M. Barr, Phys. Lett. B **112** (1982), 219; I. Antoniadis, J. R. Ellis, J. S. Hagelin and D. V. Nanopoulos, Phys. Lett. B **194** (1987), 231.

The quark and lepton superfields belong to the representation $3 \times \big(\overline{10}(1) + 5(-3) + 1(5)\big)$ of the gauge group $SU(5)\times U(1)$. Also the theory includes Higgs superfields H and $\widetilde{H}$ in $10(-1)$ and $\overline{10}(1)$; h and $\widetilde{h}$ in $5(2)$ and $\bar{5}(-2)$, and four singlets ϕ. The exact NSVZ β-functions for this model are

$$\frac{\beta_5(\alpha,\lambda)}{\alpha_5^2} = -\frac{1}{2\pi(1-5\alpha_5/2\pi)}\Big[5 + \sum_{I=1}^{3}\Big(\frac{3}{2}\gamma_{\overline{10}_I}(\alpha,\lambda) + \frac{1}{2}\gamma_{5_I}(\alpha,\lambda)\Big) + \frac{3}{2}\gamma_H(\alpha,\lambda) + \frac{3}{2}\gamma_{\widetilde{H}}(\alpha,\lambda) + \frac{1}{2}\gamma_h(\alpha,\lambda) + \frac{1}{2}\gamma_{\widetilde{h}}(\alpha,\lambda)\Big];$$

$$\frac{\beta_1(\alpha,\lambda)}{\alpha_1^2} = \frac{1}{8}\cdot\frac{1}{2\pi}\Big[60 - \sum_{I=1}^{3}\Big(2\gamma_{\overline{10}_I}(\alpha,\lambda) + 9\gamma_{5_I}(\alpha,\lambda) + 5\gamma_{E_I}(\alpha,\lambda)\Big) - 2\gamma_H(\alpha,\lambda) - 2\gamma_{\widetilde{H}}(\alpha,\lambda) - 4\gamma_h(\alpha,\lambda) - 4\gamma_{\widetilde{h}}(\alpha,\lambda)\Big].$$

D.Korneev, D.Plotnikov, K.S., N.Tereshna, ArXiv:2108.05026 [hep-th].

Three- and four-loop calculations in $\mathcal{N}=1$ supersymmetric theories made with dimensional reduction supplemented by modified minimal subtraction (i.e. in the so-called $\overline{\text{DR}}$-scheme)

L.V.Avdeev, O.V.Tarasov, Phys.Lett. **112 B** (1982) 356; I.Jack, D.R.T.Jones, C.G.North, Phys.Lett **B386** (1996) 138; Nucl.Phys. **B 486** (1997) 479; R.V.Harlander, D.R.T.Jones, P.Kant, L.Mihaila, M.Steinhauser, JHEP **0612** (2006) 024.

revealed that the NSVZ relation in the $\overline{\text{DR}}$-scheme holds only in the one- and two-loop approximations, where the β-function is scheme independent. (The NSVZ equation relates the two-loop β-function to the one-loop anomalous dimension, which is also scheme independent.)

However, in the three- and four-loop approximations it is possible to restore the NSVZ relation with the help of a specially tuned finite renormalization of the gauge coupling constant. Note that a possibility of making this finite renormalization is highly nontrivial.

This implies that the NSVZ relation holds only in some special renormalization schemes, which are usually called "NSVZ schemes", and the $\overline{\text{DR}}$-scheme is not NSVZ.

The higher covariant derivative regularization

Here we would like to derive the exact NSVZ β-function in all loops by direct summation of the perturbative series and to formulate an all-loop renormalization prescription which gives an NSVZ scheme.

The main ingredient which allows doing this is the higher covariant derivative regularization proposed by A.A.Slavnov

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301; Theor.Math.Phys. **13** (1972) 1064.

By construction, it includes the insertion of the Pauli–Villars determinants for removing residual one-loop divergencies

A.A.Slavnov, Theor.Math.Phys. **33**, (1977), 977.

Unlike dimensional reduction, this regularization is self-consistent. It can be formulated in a manifestly supersymmetric way in terms of $\mathcal{N}=1$ superfields

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745; P.West, Nucl.Phys. B268, (1986), 113.

Renormalizable non-Abelian $\mathcal{N}=1$ supersymmetric gauge theories with matter superfields at the classical level are described by the action

$$S=\frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\mbox{c.c.}\Big\}.$$

For quantizing the theory it is convenient to use the background field method. Moreover, it is necessary to take into account nonlinear renormalization of the quantum gauge superfield

O. Piguet and K. Sibold, Nucl.Phys. **B197** (1982) 257; 272;
I.V.Tyutin, Yad.Fiz. **37** (1983) 761.

This can be done with the help of the replacement $e^{2V}\to e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}$, where $\boldsymbol{V}$ and V are the background and quantum gauge superfields, respectively, and the function $\mathcal{F}(V)$ includes an infinite set of parameters needed for describing the nonlinear renormalization. The explicit form of the function $\mathcal{F}(V)$ in the lowest nontrivial order can be found in

J.W.Juer and D.Storey, Phys.Lett. **119B** (1982) 125; Nucl. Phys. **B216** (1983) 185.

For constructing the regularized theory we first add to the action terms with higher derivatives,

$$S_{\mbox{reg}} = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a\left(e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}\right)_{Adj} R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$
$$\times\left(e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\right)_{Adj} W_a + \frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\Big]_i{}^j\phi_j$$
$$+\Big[\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mbox{c.c.}\Big],$$

where the covariant derivatives are defined as

$$\nabla_a = D_a;\qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)}e^{2\boldsymbol{V}}\,\bar D_{\dot a}e^{-2\boldsymbol{V}}e^{-2\mathcal{F}(V)}.$$

Gauge is fixed by adding the term

$$S_{\mbox{gf}} = -\frac{1}{16\xi_0 e_0^2}\,\mbox{tr}\int d^4x\,d^4\theta\,\boldsymbol{\nabla}^2 V K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)_{Adj}\bar{\boldsymbol{\nabla}}^2 V.$$

Also it is necessary to introduce the Faddeev-Popov and Nielsen–Kallosh ghosts. The regulator functions $R(x)$, $F(x)$, and $K(x)$ should rapidly increase at infinity and satisfy the condition $R(0) = F(0) = K(0) = 1$.

For regularizing residual one-loop divergences and subdivergences we insert into the generating functional two Pauli–Villars determinants,

$$Z = \int D\mu\, \mathrm{Det}(PV, M_\varphi)^{-1} \mathrm{Det}(PV, M)^{c} \times \exp\Big\{i\Big(S_{\mathrm{reg}} + S_{\mathrm{gf}} + S_{\mathrm{FP}} + S_{\mathrm{NK}} + S_{\mathrm{sources}}\Big)\Big\},$$

where $D\mu$ is the functional integration measure, and

$$\mathrm{Det}(PV, M_\varphi)^{-1} \equiv \int D\varphi_1\, D\varphi_2\, D\varphi_3\, \exp(iS_\varphi);$$

$$\mathrm{Det}(PV, M)^{-1} \equiv \int D\Phi\, \exp(iS_\Phi).$$

(Here we use chiral commuting Pauli–Villars superfields.)

The superfields $\varphi_{1,2,3}$ belong to the adjoint representation and cancel one-loop divergences coming from gauge and ghost loops. The superfields Φ_i lies in a representation R_{PV} and cancel divergences coming from a loop of the matter superfields if $c = T(R)/T(R_{\mathrm{PV}})$. The masses of these superfields are

$$M_\varphi = a_\varphi \Lambda; \qquad M = a\Lambda,$$

where the coefficients a_φ and a do not depend on couplings.

It is important to distinguish renormalization group functions (RGFs) defined in terms of the bare couplings α_0 and λ_0,

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad \gamma_x(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_x}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}},$$

and RGFs standardly defined in terms of the renormalized couplings α and λ,

$$\widetilde{\beta}(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad \widetilde{\gamma}_x(\alpha,\lambda) \equiv \frac{d\ln Z_x}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}.$$

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459.

RGFs defined in terms of the bare couplings do not depend on a renormalization prescription for a fixed regularization, but depend on a regularization.

RGFs defrined in terms of the renormalized couplings depend both on regularization and on a renormalization prescription.

The HD+MSL scheme. The main statements.

Both definitions of RGFs give the same functions in the HD+MSL-scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms. This means that the renormalization constants include only powers of $\ln\Lambda/\mu$, where μ is a renormalization point.

$$\widetilde{\beta}(\alpha,\lambda)\Big|_{\mbox{HD+MSL}} = \beta(\alpha_0\to\alpha,\lambda_0\to\lambda);$$

$$\widetilde{\gamma}_x(\alpha,\lambda)\Big|_{\mbox{HD+MSL}} = \gamma_x(\alpha_0\to\alpha,\lambda_0\to\lambda).$$

Here we will briefly describe the proof of the following statements:

1. NSVZ equation is valid for RGFs defined in terms of the bare couplings in the case of using the higher covariant derivative regularization for an arbitrary renormalization prescription.

2. For RGFs defined in terms of the renormalized couplings some NSVZ schemes are given by the HD+MSL prescription. (MSL can supplement various versions of the higher covariant derivative regularization.)

The all-loop derivation of the NSVZ equation: the main steps

1. First, it is necessary to prove the ultraviolet finiteness of triple vertices with two external lines of the Faddeev–Popov ghosts and one external line of the quantum gauge superfield.

2. Next, it is necessary to rewrite the NSVZ relation in the equivalent form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

K.S., Nucl.Phys. **B909** (2016) 316.

3. After this we prove that the β-function $\beta(\alpha_0,\lambda_0)$ is determined by integrals of double total derivatives with respect to loop momenta and present a method for constructing this integrals.

K.S., JHEP **10** (2019) 011.

4. Then the NSVZ equation for RGFs defined in terms of the bare couplings is obtained by summing singular contributions.

5. Finally, an NSVZ scheme for the function $\widetilde{\beta}(\alpha,\lambda)$ is constructed.

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

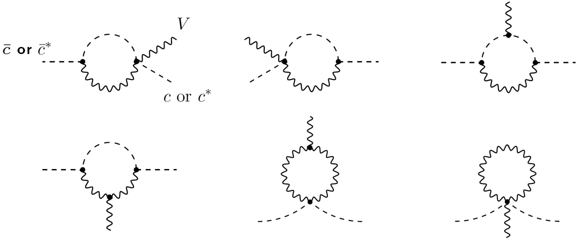
Non-renormalization of the three-point gauge-ghost vertices

The all-order finiteness of triple vertices in which two external lines correspond to the Faddeev–Popov ghosts and one external line corresponds to the quantum gauge superfield has been proved in the paper

K.S., Nucl.Phys. **B909** (2016) 316.

using the Slavnov–Taylor identities and rules for calculating supergraphs. The result is valid for the superfield formulation of the theory in the general ξ-gauge.

The one-loop contribution to these vertices comes from the superdiagrams presented below. The ultraviolet finiteness of their sum has been verified by an explicit calculation



Non-renormalization of the triple gauge-ghost vertices and the new form of the NSVZ β-function

There are 4 vertices of the considered structure, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$. All of them have the same renormalization constant $Z_\alpha^{-1/2}Z_cZ_V$. Therefore, due to their finiteness

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V) = 0,$$

where

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R; \qquad \bar{c}c = Z_c Z_\alpha^{-1}\bar{c}_R c_R.$$

The non-Abelian NSVZ equation can equivalently be rewritten as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

The β-function in the right hand side can be expressed in terms of the charge renormalization constant Z_α:

$$\beta(\alpha_0,\lambda_0) = \frac{d\alpha_0(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}.$$

Non-renormalization of the triple gauge-ghost vertices and the new form of the NSVZ β-function

Using the finiteness of the triple gauge-ghost vertices we obtain

$$\beta(\alpha_0,\lambda_0) = -2\alpha_0 \frac{d\ln(Z_c Z_V)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0) + \gamma_V(\alpha_0,\lambda_0)\Big),$$

where γ_c and γ_V are the anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare couplings), respectively.

Substituting this expression into the the right hand side we obtain the equivalent form of the NSVZ equation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

It relates the β-function in a certain loop to the anomalous dimensions of quantum superfields in the previous loop, because the right hand side does not contain a denominator depending on couplings.

The β-function of $\mathcal{N}=1$ supersymmetric gauge theories as an integral of double total derivatives

A key observation needed for derivation of the NSVZ relation is that in the case of using the higher covariant derivative regularization the integrals giving the β-function defined in terms of the bare couplings are integrals of double total derivatives in $\mathcal{N}=1$ supersymmetric gauge theories. This was first noted in

A.A.Soloshenko, K.S., ArXiv: hep-th/0304083v1 (the factorization into total derivatives);
A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445 (the factorization into double total derivatives).

The all-loop proof of this statement in the non-Abelian case has been done in

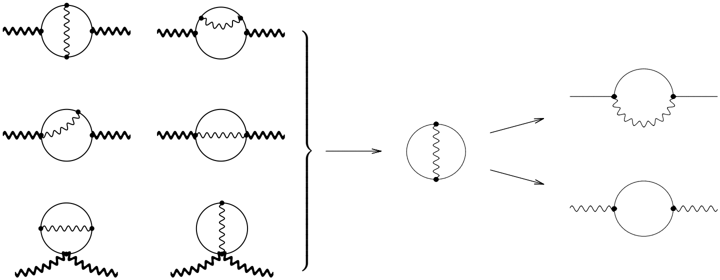
K.S., JHEP **10** (2019) 011.

The integrals of double total derivatives do not vanish due to singularities of the integrands. Really, if $f(Q^2)$ is a non-singular function which rapidly decrease at infinity, then

$$\int \frac{d^4Q}{(2\pi)^4}\frac{\partial^2}{\partial Q^\mu \partial Q_\mu}\Big(\frac{f(Q^2)}{Q^2}\Big) = \frac{1}{4\pi^2}f(0) \neq 0.$$

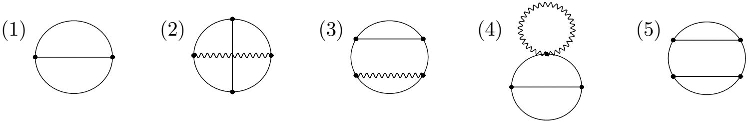
Due to similar equations the double total derivatives effectively cut internal lines. As a result we obtain diagrams contributing to various anomalous dimensions of the quantum superfields, in which a number of loops is less by 1. This allows to give a simple qualitative interpretation of the new form of the NSVZ equation:

For each vacuum supergraph the NSVZ equation relates a contribution to the β-function obtained by attaching two external lines of the background gauge superfield to the corresponding contribution to the anomalous dimension of quantum superfields obtained by all various cuts of internal lines:



An example of a certain contribution to the β-function

The two- and three-loop contributions to the β-function which depend on the Yukawa couplings are generated by the vacuum supergraphs



Here we write down the contributions of the supergraphs (1) and (5) which determine the three-loop part of the β-function quartic in the Yukawa couplings

V.Yu.Shakhmanov, K.S., Nucl.Phys., **B920**, (2017), 345;
A.E.Kazantsev, V.Yu.Shakhmanov, K.S., JHEP 1804 (2018) 130.

$$\frac{\Delta\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{2\pi}{r} C(R)_i{}^j \frac{d}{d\ln\Lambda} \int \frac{d^4K}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4}\, \lambda_0^{imn} \lambda^*_{0jmn} \frac{\partial}{\partial Q_\mu} \frac{\partial}{\partial Q^\mu} \Big(\frac{1}{K^2}$$
$$\times \frac{1}{F_K\, Q^2 F_Q\, (Q+K)^2 F_{Q+K}}\Big) + \frac{4\pi}{r} C(R)_i{}^j \frac{d}{d\ln\Lambda} \int \frac{d^4K}{(2\pi)^4} \frac{d^4L}{(2\pi)^4} \frac{d^4Q}{(2\pi)^4} \Big[\lambda_0^{iab}$$
$$\times \lambda^*_{0kab} \lambda_0^{kcd} \lambda^*_{0jcd} \Big(\frac{\partial}{\partial K_\mu}\frac{\partial}{\partial K^\mu} - \frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big) + 2\lambda_0^{iab} \lambda^*_{0jac} \lambda_0^{cde} \lambda^*_{0bde} \frac{\partial}{\partial Q_\mu}\frac{\partial}{\partial Q^\mu}\Big]$$
$$\times \frac{1}{K^2 F_K^2\, Q^2 F_Q\, (Q+K)^2 F_{Q+K}\, L^2 F_L\, (L+K)^2 F_{L+K}} = -\frac{1}{2\pi r} C(R)_i{}^j (\Delta\gamma_\phi)_j{}^i.$$

The result for these supergraphs (as well for the other ones presented in the previous slide) exactly agrees with the qualitative interpretation of the NSVZ equation.

Thus, all higher order corrections to the β-function turn out to be determined by singular contributions which appear in calculating the integrals of double total derivatives. In all orders of the perturbation theory sums of singular contributions which come from cuts of various propagators have been calculated in

K.S., Eur.Phys.J. **C80** (2020) 10, 911.

The result of this calculation can be written as

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} - \frac{\beta_{\textbf{1-loop}}(\alpha_0)}{\alpha_0^2}$$

$$= \underbrace{\frac{1}{\pi}C_2\gamma_V(\alpha_0,\lambda_0)}_{\text{gauge propagators}} + \underbrace{\frac{1}{\pi}C_2\gamma_c(\alpha_0,\lambda_0)}_{\text{Faddeev–Popov ghost propagators}} - \underbrace{\frac{1}{2\pi r}C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)}_{\text{matter propagators}}.$$

Substituting the expression for the one-loop β-function we obtain the main result:

The NSVZ relation

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0) -2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big),$$

and, therefore, the NSVZ relation

$$\beta(\alpha_0,\lambda_0) = -\frac{\alpha_0^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big)}{2\pi(1-C_2\alpha_0/2\pi)}$$

are valid in all orders of the perturbation theory for RGFs defined in terms of the bare couplings if a theory is regularized by higher covariant derivatives.

Consequently, for RGFs defined in terms of the renormalized couplings, similar equations hold in the HD+MSL scheme in all orders of the perturbation theory.

Conclusion

- In the case of using the regularization by higher covariant derivatives RGFs defined in terms of the bare couplings satisfy the NSVZ equation in all orders for any renormalization prescription.
- RGFs defined in terms of the renormalized couplings satisfy the NSVZ equation in the HD+MSL scheme, when a theory is regularized by higher covariant derivatives, and divergences are removed by minimal subtractions of logarithms.
- The β-function of $\mathcal{N}=1$ supersymmetric gauge theories is determined by integrals of double total derivatives in the momentum space if a theory is regularized by higher covariant derivatives.
- The triple gauge-ghost vertices are UV finite in all orders. This allows to rewrite the NSVZ relation in an equivalent form, which relates the β-function to the anomalous dimensions of the quantum superfields.

Thank you for the attention!