Obtaining fully polarised amplitudes in gauge invariant form

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In collaboration with **Christian Schubert** (UMSNH), **Olindo Corradini** (Modena), **Naser Ahmadiniaz** (Helmholz Zentrum, Dresden), **Fiorenzo Bastianelli** (Bologna), **Victor Guzmán** (UMSNH) and others.

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Outline

1. Introduction
   - Scattering in QFT
   - Amplitude calculations
2. Worldline approach
   - Worldline representation
   - Amplitudes
3. Overview

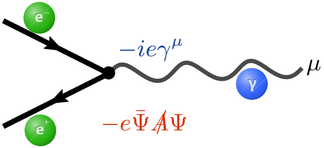
Introduction

Quantum Field theory

We will work within standard QED: a Dirac field coupled to a gauge potential $\mathrm{A}(x) = A_\mu(x) dx^\mu$ described by the familiar Dirac action with *"minimal coupling"*

$$S[\Psi, A] = \int d^D x \left[-\frac{1}{4}\mathrm{tr}F^2 + \bar{\Psi}\big(i\not{D} - m\big)\Psi \right]$$

where the covariant derivative is $D_\mu := \partial_\mu + ieA_\mu$, which couples the fields together and produces the interaction vertex of the quantum theory



We can interpret "particles" as excitations of the quantum field about *"the vacuum,"* a state of minimal energy denoted by $|0\rangle$:

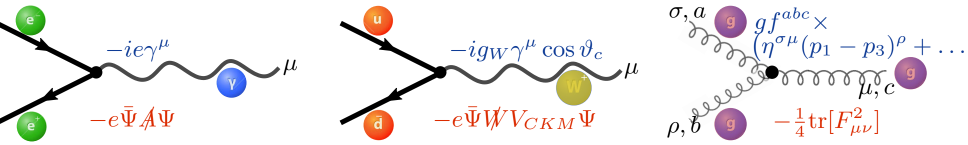
$$|p, \sigma\rangle = \widehat{a}^\dagger_{p,\sigma}|0\rangle \qquad\qquad |k, h\rangle = \widehat{\alpha}^\dagger_{k,h}|0\rangle$$

Interactions

Working with canonical quantisation in the interaction picture the fundamental objects of interest are **correlation functions**:

$$\langle\Omega|T\{\hat{\Psi}(x_1)\ldots\hat{\bar{\Psi}}(x_n)\ldots\hat{A}_{\mu_N}(x_N)\}|\Omega\rangle\,.$$

In standard **perturbation theory** we split the action into a "free" part and an "interaction" part which couples the fields together – these indicate the interaction vertices of the theory. Some examples from the **standard model** include:



We use the above vertices to form Feynman diagrams that contribute order by order in the coupling constants:

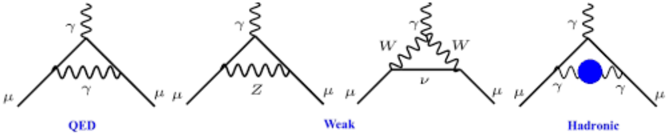


Diagram complexity

Perturbative calculations of amplitudes often involve tedious manipulation of long expressions....

But lead to surprisingly simple final results:

> **Weisskopf** *was so unhappy with the conventional calculation of the Compton cross section that he asked me, and several other Ph.D. students, to find a better way to get the* **simple final result**. *(R. Stora).*

There is also a **factorial explosion** in the number of diagrams at a given loop order that makes organisation of the calculation progressively more difficult.

Example: electron $g-2$ (QED):

- **1** diagram at one-loop order: $\frac{g-2}{2}+ = \frac{1}{2}\frac{\alpha}{\pi}$ (Schwinger[1])
- **7** diagrams at two-loops: $\frac{g-2}{2}+ = -0.328\ldots\left(\frac{\alpha}{\pi}\right)^2$ (Petermann / Sommerfeld[2])
- **72** diagrams at three-loops: $\frac{g-2}{2}+ = 1.181\ldots\left(\frac{\alpha}{\pi}\right)^3$ (Laporta, Remiddi[3])
- **891** diagrams at 4-loops: $\frac{g-2}{2}+ = -1.912\ldots\left(\frac{\alpha}{\pi}\right)^4$ (Laporta[4])
- **12672** diagrams at 5-loop order (some numerical evaluations)

[1] Schwinger, J.S. Phys. Rev. 73, (1948), 416–417.
[2] Petermann, Helv. Phys. Acta 30, (1957), 407–408 & Sommerfield, C.M. Ann. Phys. 5 (1958), 26–57
[3] Laporta, S., Remiddi, E. Phys. Lett. B 379, (1996), 283–291
[4] Laporta, S. Phys. Lett. B 772 (2017), 232–238

Miraculous cancellations

Natural question: *¿Why are these final results so small?*
Aside from this, for $g-2$ there is also a cancellation of spurious UV and IR divergences between diagrams[5] due to gauge symmetry.

[5] See G. V. Dunne et al., J. Phys. Conf. Ser.37, 59–72 (2006) for a discussion on the influence of gauge cancellations on divergence structure in QFT which is still not very well understood.

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Is this gauge invariance also responsible for the spectacular cancellations of the finite part of the amplitude?

Motivated P. Cvitanović to examine the contributions from individual "**gauge-sets**" – sets of gauge invariant diagrams. The contributions of different gauge sets are very close to being **integer multiples** of $\pm\frac{1}{2}\times\left(\frac{\alpha}{\pi}\right)^n$!

Cvitanović also made the conjecture that this suggests that the perturbative series for $g-2$ may in fact converge, having asymptotic form

$$\frac{g-2}{2}=\sum_{n=1}^{\infty}c_n\Big(\frac{\alpha}{\pi}\Big)^{2n}\,, \qquad |c_n|\sim\frac{n}{2}\,, \tag{1}$$

rather than the usual factorial growth.

[5]See G. V. Dunne et al., J. Phys. Conf. Ser.37, 59–72 (2006) for a discussion on the influence of gauge cancellations on divergence structure in QFT which is still not very well understood.

Worldline approach

The worldline representation

The **worldline formalism**[6] is an alternative method for the quantising fields, based on the *first quantisation* of relativistic particles.

In this worldline approach to field theory the dressed electron propagator has path integral representation ($D_\mu = \partial_\mu + ieA_\mu$)

$$S^{x'x} = \langle x' \big| \big[m - i\not{D}\big]^{-1} \big| x \rangle = \big[m + i\not{D}\big] \langle x' \big| \big[m^2 - D^2 + \frac{ie}{2}\gamma^\mu F_{\mu\nu}\gamma^\nu\big]^{-1} \big| x \rangle$$

$$\equiv \big[m + i\not{D}\big] K^{x'x}[A]$$

[6]Phys. Rev. **E80**, 3 (1950), 440, [7]Nucl. Phys. **B385**, (1992), 145

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$$\equiv \big[m + i\not{D}\big] K^{x'x}[A]$$

Here we defined the **kernel** $K^{x'x}$ whose path integral / proper-time representation is

$$K^{x'x} = 2^{-\frac{D}{2}} \mathrm{symb}^{-1} \int_0^\infty dT\, \mathrm{e}^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathscr{D}x \int_{APC} \mathscr{D}\psi\, \mathrm{e}^{-S[x,\psi]} \tag{2}$$

with the **worldline action** of a minimally coupled relativistic point particle

$$S[x,\psi] = \int_0^T d\tau \left[\frac{\dot{x}^2}{4} + \frac{i}{2}\psi \cdot \dot{\psi} + eA(x) \cdot \dot{x} + ie(\psi + \eta)^\mu F_{\mu\nu}(x)(\psi + \eta)^\nu \right].$$

on which the **symbol map** acts according to $\mathrm{symb}(\gamma^{\alpha_1 \cdots \alpha_n}) = (-i\sqrt{2})^n \eta^{\alpha_1} \cdots \eta^{\alpha_n}$.

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Scattering Amplitudes

For N-**photon** scattering amplitudes we decompose the background into plane waves,

$$A^\mu(x) = \sum_{i=0}^{N} \varepsilon_i^\mu \mathrm{e}^{ik_i\cdot x} .$$

Interactions with photons are represented by **vertex operators** under the path integral:

$$V_\eta^{x'x}[k,\varepsilon] = \int_0^T d\tau \left[\varepsilon\cdot\dot{x} + 2ie\varepsilon\cdot(\psi+\eta)k\cdot(\psi+\eta)\right]\mathrm{e}^{ik\cdot x}$$

similarly to in string perturbation theory.

In **momentum space** we recover the photon-dressed propagator from

$$S_N^{p'p}[\{k_i,\varepsilon_i\}] = (\not{p}' + m)K_N^{p'p}[\{k_i,\varepsilon_i\}] - e\sum_{j=1}^{N} \not{\varepsilon}_j K_{N-1}^{p'+k_j,p}[\{k_i,\varepsilon_i\},\hat{k}_j,\hat{\varepsilon}_j] \qquad (3)$$

which splits into parts we call leading and subleading

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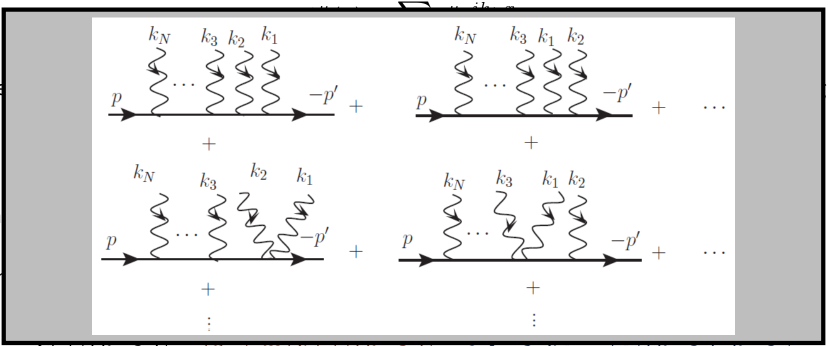
Inte ... ral:



sim

In r

$$S_N[\{k_i,\varepsilon_i\}] = (\not p + m)K_N[\{k_i,\varepsilon_i\}] - e\sum_{j=1} \not\varepsilon_j K_{N-1}[\{k_i,\varepsilon_i\},k_j,\varepsilon_j] \quad (3)$$

which splits into parts we call leading and subleading

Gauge invariance

To manifest the gauge invariance of the *on-shell* scattering amplitudes – which must satisfy the QED **Ward identity** – we rewrite the vertex operator as

$$
\begin{aligned}
V_\eta^{x'x}[k,\varepsilon] &= \int_0^T d\tau \, \Big[\varepsilon \cdot \dot{x} + i \frac{\varepsilon \cdot r}{k \cdot r} \frac{d}{d\tau} - ie(\psi + \eta) \cdot f \cdot (\psi + \eta)\Big] \mathrm{e}^{ik\cdot x} \\
&= \int_0^T d\tau \, \Big[\frac{r \cdot f \cdot \dot{x}}{r \cdot k} - ie(\psi + \eta) \cdot f \cdot (\psi + \eta)\Big] \mathrm{e}^{ik\cdot x}
\end{aligned}
$$

where introduce an auxiliary "reference vector" r (the boundary terms will vanish when the LSZ formula is applied as they will be missing the required pole structure).

In the on-shell limit, only the leading term in the propagator contributes to the S-matrix (here we "amputate" the external fermion legs):

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In the on-shell limit, only the leading term in the propagator contributes to the S-matrix (here we "amputate" the external fermion legs):

$$
\begin{aligned}
\mathcal{M}_N^{p'p} &= \bar{u}(-p')(-\not{p}+m)(\not{p}+m)S_N^{p'p}(\not{p}+m)u(p) \\
&= \bar{u}(-p')\frac{\mathfrak{K}_N^{p'p}}{2m}u(p)\,.
\end{aligned}
$$

which will now be a function of the field strength tensors of the individual photons.

Simple examples

The kernel admits a natural decomposition onto the even sub-algebra of the Dirac representation of the Clifford algebra (see arXiv:2004.01391):

$$K_N^{p'p} \equiv (-ie)^N \frac{\mathfrak{K}_N^{p'p}}{(p'^2+m^2)(p^2+m^2)}\,, \quad \mathfrak{K}_N^{p'p} \equiv A_N \mathbb{1} + B_{N\alpha\beta}\sigma^{\alpha\beta} - iC_N\gamma_5\,.$$

Example: **N=1** (vertex):

$$A_1 = -2\frac{r\cdot f\cdot p}{r\cdot k}\,, \qquad B_{1\alpha\beta} = \frac{1}{2}f_{\alpha\beta}$$

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$$A_1 = -2\frac{r\cdot f\cdot p}{r\cdot k}\,, \qquad B_{1\alpha\beta} = \frac{1}{2}f_{\alpha\beta}$$

Example: **N=2** (Compton):

$$A_2 = -2\frac{r_1\cdot f_1\cdot f_2\cdot r_2}{r_1\cdot k_1 r_2\cdot k_2} - \frac{1}{2}\left[\frac{1}{2p'\cdot k_1} + \frac{1}{2p'\cdot k_2}\right]\mathrm{tr}\big(f_1\cdot f_2\big)$$

$$B_2^{\alpha\beta} = -\frac{1}{2}\frac{r_1\cdot f_1\cdot k_2 f_2^{\alpha\beta} + r_2\cdot f_2\cdot k_1 f_1^{\alpha\beta}}{r_1\cdot k_1 r_2\cdot k_2} + \frac{1}{2}\left[\frac{1}{2p'\cdot k_1} - \frac{1}{2p'\cdot k_2}\right][f_1, f_2]^{\alpha\beta}$$

$$C_2 = \frac{1}{2}\left[\frac{1}{2p'\cdot k_1} + \frac{1}{2p'\cdot k_2}\right]\mathrm{tr}\big(f_1\cdot \tilde{f}_2\big)\,, \qquad (\tilde{f}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}f_{\alpha\beta})\,.$$

Benefits of worldline approach

Using our first quantised representation we gain certain advantages over standard techniques:

1. Simple formula for **unpolarised** amplitudes: $\left\langle |\mathcal{M}_N|^2 \right\rangle = e^{2N}\left[|A_N|^2 + 2B_N^{\alpha\beta}B^{\star}_{N\alpha\beta} - |C_N|^2\right]$
2. Manifest gauge invariance of the on-shell amplitudes, expressed in terms of **field strength tensors** at the level of the (worldline) integrand.
3. Allows a complete separation of photon and electron **polarisations**, conveniently formulated in the spinor helicity framework.
4. Facilitates a "spin-orbit decomposition" that differentiates the photons' interactions with the spin and orbital degrees of freedom of the electron.
5. Simple proofs of the vanishing of massless equal-helicity amplitudes in the *massless* case (c.f. Mahlon Phys.Rev.D 49 (1994) 2197)...
6. ...and unexpected relations between these amplitudes in scalar and spinor QED for the *massive* case.
7. Relations between the coefficients A_N, $B_{N\alpha\beta}$ and C_N, and recursion relations relating these to their $N-1$ counterparts.

Overview

Conclusion

First quantised representations of field theory processes can be powerful alternatives to standard techniques that can make certain structural aspects of the theory clearer. *See following talk by* José Nicasio!

We have given a worldline description of the dressed electron propagator and produced a generating function for N-photon amplitudes that is manifestly transverse.

Future work:

1. Extend the first quantised representation to propagation in the presence of electromagnetic or gravitational background fields
2. Generalise to the non-Abelian case using "worldline colour fields" – Master Formula for gluon amplitudes
3. Apply to study higher-order processes, such as to calculation of the electron's anomalous gyromagnetic moment.

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For more information on worldline techniques see

- Classic review: C. Schubert Phys. Rept. 355 (2001) 73 [arXiv:0101036 [hep-th]]
- More up-to-date report: **JPE** and C. Schubert [arXiv:1912.10004 [hep-th]]

¡Thank you for your attention!