Probing nonlinear electrodynamics with a single superconducting radio-frequency cavity

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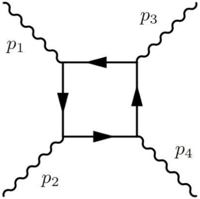
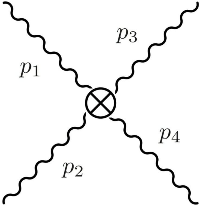
Nonlinear electrodynamics (Euler-Heisenberg)

Interactions with virtual electrons are integrated out (the limit $p \ll m_e$),





The Euler-Heisenberg effective Lagrangian[1]

$$\mathcal{L} = -\frac{1}{4}\mathcal{F} + \varepsilon\left(\mathcal{F}^2 + \frac{7}{4}\mathcal{G}^2\right), \qquad \varepsilon = \frac{\alpha_e^2}{90 m_e^4}.$$

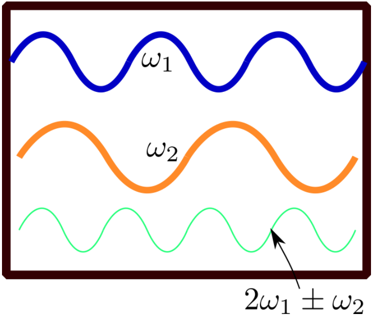
$$\mathcal{F} = F_{\mu\nu}F^{\mu\nu} = -2\left(\boldsymbol{E}^2 - \boldsymbol{H}^2\right), \qquad \mathcal{G} = F_{\mu\nu}\widetilde{F}^{\mu\nu} = -4\left(\boldsymbol{E}\cdot\boldsymbol{H}\right),$$

[1] H.Euler and B. Kockel (1935), W.Heisenberg and H.Euler (1936)

Nonlinear interactions in cavities

Nonlinear theory with 4-photon interaction $\rightarrow$ $\gamma\gamma \rightarrow \gamma\gamma$ scattering
Radio modes in cavities: possibility of generation of modes with combined frequency[2]

- pump modes: ω_1, ω_2
- signal modes: $2\omega_{1(2)} \pm \omega_{2(1)}$, $3\omega_{1(2)}$.
- Resonant amplification of a signal mode
- $Q \sim 10^{11}$ in SRF cavities



- The authors[2] have not presented the resonant solution explicitly.
- No generation of the 3rd harmonics in 1d! K. Shibata. (2020)
- Our goal is to present explicit solution 1d and 3d

[2] G. Brodin, M. Marklund, L. Stenflo. PRL (2001), D. Eriksson, G. Brodin, M. Marklund, L. Stenflo. Phys. Rev. A (2004)

Nonlinear Maxwell and wave equations

Modified Maxwell equations

$$\mathrm{rot}\,\boldsymbol{H} = \frac{\partial \boldsymbol{E}}{\partial t} + \left[\frac{\partial \boldsymbol{P}}{\partial t} - \mathrm{rot}\,\boldsymbol{M}\right]$$

$$\mathrm{rot}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\mathrm{div}\,\boldsymbol{E} = [-\,\mathrm{div}\,\boldsymbol{P}]$$

$$\mathrm{div}\,\boldsymbol{H} = 0$$

Vacuum polarization and magnetisation vectors

$$\boldsymbol{P} = 16\varepsilon\left[\left(E^2 - H^2\right)\boldsymbol{E} + 7/2(\boldsymbol{E}\cdot\boldsymbol{H})\boldsymbol{H}\right]$$

$$\boldsymbol{M} = 16\varepsilon\left[\left(E^2 - H^2\right)\boldsymbol{H} - 7/2(\boldsymbol{E}\cdot\boldsymbol{H})\boldsymbol{E}\right]$$

Modified wave equations

$$\Box\boldsymbol{E} = \frac{\partial}{\partial t}\,\mathrm{rot}\,\boldsymbol{M} + \mathrm{grad}\,\mathrm{div}\,\boldsymbol{P} - \frac{\partial^2 \boldsymbol{P}}{\partial t^2}$$

$$\Box\boldsymbol{H} = \frac{\partial}{\partial t}\,\mathrm{rot}\,\boldsymbol{P} - \mathrm{grad}\,\mathrm{div}\,\boldsymbol{M} + \Delta\boldsymbol{M}$$

Resonant amplification of a signal mode

EM field: *pump* mode (initially given) + *signal* mode (we are looking for it), $\boldsymbol{E} = \boldsymbol{E}^p + \boldsymbol{E}^{sig}$, $\boldsymbol{H} = \boldsymbol{H}^p + \boldsymbol{H}^{sig}$

Hierarchy: $\boldsymbol{E}^{sig} \sim \varepsilon \left(\boldsymbol{E}^p\right)^3 \ll \boldsymbol{E}^p$.

Perturbative equations

$$\Box \boldsymbol{E}^{sig} = \frac{\partial}{\partial t} \operatorname{rot} \boldsymbol{M}(\boldsymbol{E}^p, \boldsymbol{H}^p) + \operatorname{grad} \operatorname{div} \boldsymbol{P}(\boldsymbol{E}^p, \boldsymbol{H}^p) - \frac{\partial^2 \boldsymbol{P}(\boldsymbol{E}^p, \boldsymbol{H}^p)}{\partial t^2},$$
$$\Box \boldsymbol{H}^{sig} = \frac{\partial}{\partial t} \operatorname{rot} \boldsymbol{P}(\boldsymbol{E}^p, \boldsymbol{H}^p) - \operatorname{grad} \operatorname{div} \boldsymbol{M}(\boldsymbol{E}^p, \boldsymbol{H}^p) + \Delta \boldsymbol{M}(\boldsymbol{E}^p, \boldsymbol{H}^p)$$

may have resonantly growing solutions,

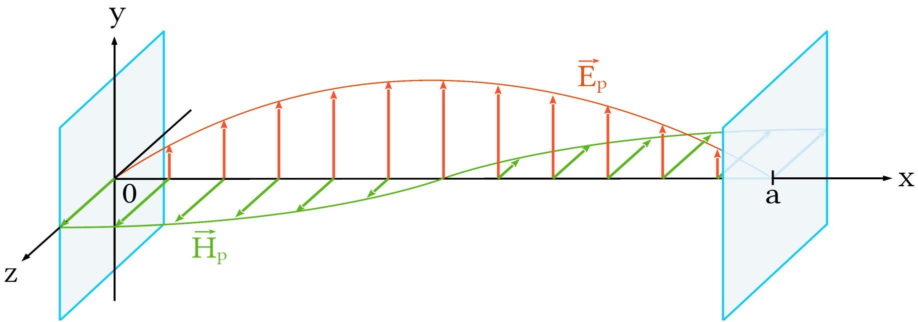
$$\Box E_x = \cos(\omega_r t) \sin(\omega_r x) + \ldots \quad \to \quad E_x^{(growing)} = \frac{t}{\omega_r} \sin(\omega_r t) \sin(\omega_r x).$$

Introduce dissipation Γ:

$$(\Box + \Gamma \partial_t) E_x = \cos(\omega_r t) \sin(\omega_r x) \quad \to \quad E_x^{(steady)} = \frac{1}{\Gamma \omega_r} \sin(\omega_r t) \sin(\omega_r x).$$

Single pump mode in one-dimensional cavity

$L_y, L_z \gg L_x \equiv a$



Pump mode: $\left(\omega_n = k_n = n \cdot \frac{\pi}{a}\right), \; n \in \mathbb{N}.$

$$\begin{cases} \boldsymbol{E}^{pump}(x,t) = F_0 \cdot \sin(k_n x) \sin(\omega_n t) \cdot \boldsymbol{e}_y, \\ \boldsymbol{H}^{pump}(x,t) = F_0 \cdot \cos(k_n x) \cos(\omega_n t) \cdot \boldsymbol{e}_z. \end{cases}$$

Single pump mode in one-dimensional cavity

$L_y, L_z \gg L_x \equiv a$

Pump mode: $E_y^{pump} = F_0 \cdot \sin(\omega_n x)\sin(\omega_n t), \qquad H_z^{pump} = F_0 \cdot \cos(\omega_n x)\cos(\omega_n t).$

EOMs for signal modes[3]:

$$(\Box + \Gamma\partial_t)\, E_y^{sig} = 8\varepsilon F_0^3 \omega_n^2 \Big[2\sin(\omega_n x)\sin(\omega_n t) - 3\sin(\omega_n x)\sin(3\omega_n t) + \sin(3\omega_n x)\sin(\omega_n t)\Big],$$

$$(\Box + \Gamma\partial_t)\, H_z^{sig} = 8\varepsilon F_0^3 \omega_n^2 \Big[2\cos(\omega_n x)\cos(\omega_n t) - \cos(\omega_n x)\cos(3\omega_n t) + 3\cos(3\omega_n x)\cos(\omega_n t)\Big].$$

wavenumbers in RHS	n	$3n$
corresponding frequencies	$\omega_n,\ \omega_{3n}$	ω_n

Evolution of the signal mode[4]:

- The frequency component ω_n is resonantly amplified
  But: hard to distinguish from the pump mode
- No resonant amplification for the triple frequency $3\omega_n$.

[3] Analytical calculations made in "wxMaxima 21.02.0" computer algebra system.
[4] In agreement with *K. Shibata. EPJ D (2020)*.

Two pump modes in one-dimensional cavity

$L_y, L_z \gg L_x \equiv a$

Two pump modes with freq-s: $\omega_n = n \cdot \frac{\pi}{a}, \ \omega_p = p \cdot \frac{\pi}{a}, \quad n, p \in \mathbb{N},$

$$\begin{cases} \boldsymbol{E}^{pump}(x,t) = F_0 \operatorname{Re}\left[\boldsymbol{\mathcal{E}}_n(x) \cdot e^{i\omega_n t} + \hat{\boldsymbol{R}}_x(\alpha)\boldsymbol{\mathcal{E}}_p(x) \cdot e^{i\omega_p t}\right], \\ \boldsymbol{H}^{pump}(x,t) = F_0 \operatorname{Re}\left[\boldsymbol{\mathcal{M}}_n(x) \cdot e^{i\omega_n t} + \hat{\boldsymbol{R}}_x(\alpha)\boldsymbol{\mathcal{M}}_p(x) \cdot e^{i\omega_p t}\right], \end{cases} \qquad \hat{\boldsymbol{R}}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

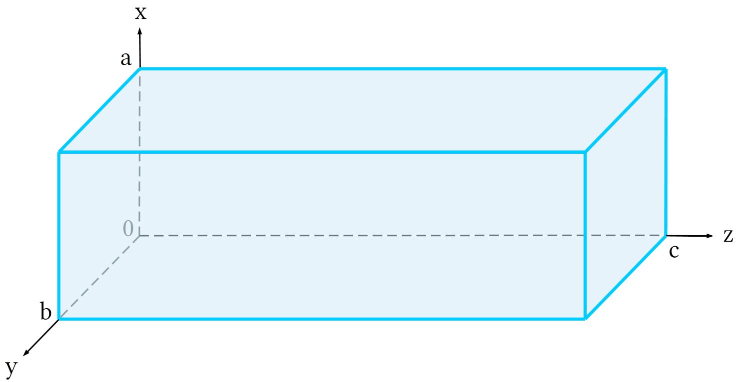
RHS for the EOMs for signal modes:

wavenumbers	n	$3n$	$2n-p$	$2n+p$
corr. freq-s.	ω_n, $\omega_{2p\pm n}$, ω_{3n}	ω_n	ω_p, ω_{2n+p}	ω_p, ω_{2n-p}

$+ \ (n \leftrightarrow p)$

- The only resonant components are ω_n and ω_p
- No terms like $\cos((2\omega_n \pm \omega_p)t)\cos((2\omega_n \pm \omega_p)x)$ in the RHS
- The mixed frequency components $\omega_{2n\pm p}$ are not resonant!

3d rectangular cavity: single pump mode.



Pump modes in 3d rectangular cavitiy:

$$\begin{array}{lll} \boldsymbol{\mathcal{E}}^{TM}_{npq}(\boldsymbol{r}), & \boldsymbol{\mathcal{M}}^{TM}_{npq}(\boldsymbol{r}), & n,p \in \mathbb{N},\ q \in \mathbb{N}_0 \\ \boldsymbol{\mathcal{E}}^{TE}_{npq}(\boldsymbol{r}), & \boldsymbol{\mathcal{M}}^{TE}_{npq}(\boldsymbol{r}), & n,p \in \mathbb{N}_0,\ q \in \mathbb{N} \end{array}, \quad \omega_{npq} = \pi\sqrt{\frac{n^2}{a^2} + \frac{p^2}{b^2} + \frac{q^2}{c^2}}.$$

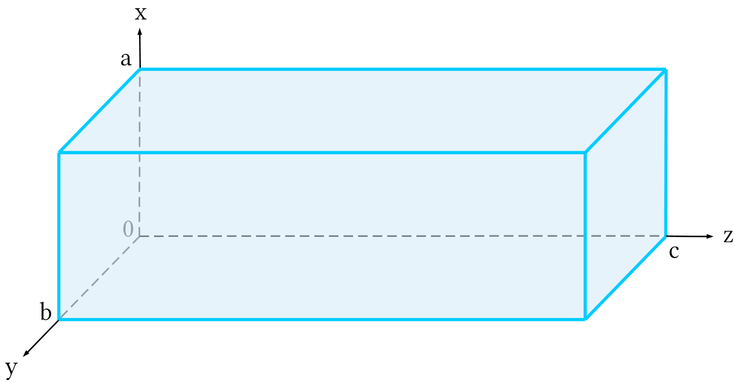
3d rectangular cavity: single pump mode.

Pump mode: ω_{npq}.
RHS for the EOMs for signal modes:

wn:	n, p, q	$3n, p, q$	$3n, 3p, q$	$(n \leftrightarrow p \leftrightarrow q)$	$3n, 3p, 3q$
freq	$\omega_{npq},\ 3\omega_{npq}$				ω_{npq}

- The pump mode frequency component ω_{npq} is resonantly amplified
- The triple frequency component $3\omega_{npq}$ is not resonantly amplified

3d rectangular cavity: two pump modes.



Pump modes: $TM_{110} + TE_{011}$

$$\boldsymbol{E}^{pump}(\boldsymbol{x},t) = F_0 \cdot \mathrm{Re}\left[\boldsymbol{\mathcal{E}}_{110}^{TM}(\boldsymbol{x}) \cdot e^{i\omega_{110}t} + \boldsymbol{\mathcal{E}}_{011}^{TE}(\boldsymbol{x}) \cdot e^{i\omega_{011}t}\right],$$

$$\boldsymbol{H}^{pump}(\boldsymbol{x},t) = F_0 \cdot \mathrm{Re}\left[\boldsymbol{\mathcal{M}}_{110}^{TM}(\boldsymbol{x}) \cdot e^{i\omega_{110}t} + \boldsymbol{\mathcal{M}}_{011}^{TE}(\boldsymbol{x}) \cdot e^{i\omega_{011}t}\right].$$

3d rectangular cavity: two pump modes.

Pump modes: $TM_{110} + TE_{011}$

Signal modes:

TM-modes:	**110**	130	310	330
	$\boldsymbol{\omega_{110}}$, $3\omega_{110}$, $2\omega_{011} \pm \omega_{110}$		ω_{110}, $3\omega_{110}$	ω_{110}
TM-modes:	112	132	211	231
	ω_{110}, $2\omega_{011} \pm \omega_{110}$		ω_{011}, $2\omega_{110} \pm \omega_{011}$	
TE-modes:	**011**	031	013	033
	$\boldsymbol{\omega_{011}}$, $3\omega_{011}$, $2\omega_{110} \pm \omega_{011}$		ω_{011}, $3\omega_{011}$	ω_{011}
TE-modes:	112	132	211	231
	ω_{110}, $2\omega_{011} \pm \omega_{110}$		ω_{011}, $2\omega_{110} \pm \omega_{011}$	

- ω_{110} (ω_{011}) resonate, $3\omega_{110}$ ($3\omega_{011}$) not resonate
- For combined frequency check the freq. condition $2\omega_{011} \pm \omega_{110} = \omega_{130}$
- No resonance for the freq. sum $2\omega_{011} + \omega_{110}$

3d rectangular cavity: two pump modes

$$2\omega_{011} - \omega_{110} = \omega_{130}$$

$$\omega_{npq} = \pi\sqrt{\frac{n^2}{a^2} + \frac{p^2}{b^2} + \frac{q^2}{c^2}} \qquad \Rightarrow \qquad \left(\frac{b}{c}\right)^2 = \frac{3\left(\frac{a}{c}\right)^2 + 1}{\left(\frac{a}{c}\right)^2 - 1}$$

$$a : b : c = \xi : \xi : 1, \qquad \xi = \sqrt{\frac{3\xi^2 + 1}{\xi^2 - 1}} \approx 2.058$$

The signal component E_z:

$$E_z(\boldsymbol{x}, t) = \frac{\pi^2 F_0^3 \varepsilon}{c^2 \Gamma} \cdot \left[\frac{A}{\omega_{110}} \sin(\omega_{110} t) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) + \right.$$

$$\left. + \frac{B}{\omega_{130}} \sin(\omega_{130} t) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right)\right] + \ \ldots, \qquad \begin{array}{l} A \approx 17.17, \\ B \approx 8.517, \end{array}$$

The amplitude: $\quad E_z^0 = \dfrac{\pi^2 F_0^3 Q \varepsilon}{c^2} \dfrac{B}{\omega_{130}^2} \approx 0.2\ \dfrac{\mu V}{\mathrm{m}}$

Conclusions

- Third harmonics ($3\omega_1$) is not resonantly amplified in 1d and 3d rectangular cavity
- Modes with combined frequency ($2\omega_1 \pm \omega_2$) are not resonantly amplified in 1d
- Mode ($2\omega_1 - \omega_2$) is resonantly amplified for a certain geometry. Explicit solution for $\omega_{130} = 2\omega_{011} - \omega_{110}$
- Mode ($2\omega_1 + \omega_2$) – no resonance for pump modes $TM_{110} + TE_{011}$
  for general set of modes – in progress..

Thank you for your attention!

General resonance criterium

Cavity D with border S. Signal field $f = E_x^{sig}, E_y^{sig}, ...H_z^{sig}$.

$$\begin{cases} (\Box + \Gamma\Omega \cdot \partial_t)\, f(M,t) = F(M,t), & M \in D,\ t > 0, \\ f(M,0) = 0, & M \in D, \\ \left(a + b\dfrac{\partial}{\partial \vec{n}}\right) f(P) = 0, & P \in S. \end{cases}$$

$$f(M,t) = \sum_{k=1}^{\infty} f_k(t) \cdot \mathcal{E}_k(M).$$

$$\ddot{f}_n(t) + \Gamma\Omega \dot{f}_n(t) + \omega_n^2 f_n(t) = F_n(t), \quad F_n(t) \equiv \frac{(F(\cdot,t),\ \mathcal{E}_n)}{\|\mathcal{E}_n\|^2}.$$

Resonance criterioum for the signal frequency ω_{sig}

1. ω_{sig} belongs to the cavity spectrum ($\exists\ m \in \mathbb{N}:\ \omega_{sig} = \omega_m$),
2. Temporal spectrum $F_m(t)$ consists ω_{sig}.